

MATHEMATICS

(Honours Core)

Paper : MAT-HC-4016

(Multivariate Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed :
1 × 10 = 10

- (a) If $f(x, y) = \ln(y - x)$, then find domain of it.
- (b) Define level curve of a function $f(x, y)$ at a constant C .
- (c) Find f_x if $f(x, y) = (\sin x^2) \cos y$.

Contd.

(d) If $f(x, y) = \sin xy$, then df is.

- (i) $y \cos xy dx + x \cos xy dy$
- (ii) $y \cos xy dy + x \cos xy dx$
- (iii) $y \cos x dx + x \cos y dy$
- (iv) $\cos xy dx + \cos xy dy$

(Choose the correct option)

(e) Evaluate $\frac{\partial(x, y)}{\partial(r, \theta)}$ for the transformation

$$x = r \cos \theta, y = r \sin \theta.$$

(f) If $P_0(x_0, y_0)$ is a critical point of $f(x, y)$ and f has continuous 2nd order partial derivatives in a disk centered at (x_0, y_0) and $D = f_{xx}f_{yy} - f_{xy}^2$, then a relative minimum occurs at P_0 , if

- (i) $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$
- (ii) $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$
- (iii) $D(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) < 0$
- (iv) $D(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) > 0$

(Choose the correct option)

(g) The curl of a vector field

$$V(x, y, z) = u(x, y, z)i + v(x, y, z)j + w(x, y, z)k \text{ is}$$

- (i) $\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) k$
- (ii) $\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) k$
- (iii) $\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) i + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) j + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) k$
- (iv) $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) i + \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) j + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) k$

(Choose the correct option)

(h) Define work as a line integral.

(i) State Green's theorem on a simply connected region D .

(j) If a vector field F and curl F are both continuous in a simply connected region D on \mathbb{R}^3 , then F is conservative in D if $\text{curl } F \neq 0$. State whether this statement is true or false.

2. Answer the following questions :

2×5=10

(a) Show that the function f is continuous at $(0, 0)$ where

$$f(x, y) = \begin{cases} \frac{y \sin \frac{1}{x}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(b) Compute the slope of the tangent line to the graph of $f(x, y) = x^2 \sin(x + y)$ at the point $P\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ in the direction parallel to the XZ plane.

(c) Evaluate $\iint_R \frac{2xy}{x^2+1} dA$ where $0 \leq x \leq 1$, $1 \leq y \leq 3$.

(d) Show that $\text{div } F = 0$ and $\text{curl } F = 0$, if F is a constant vector field.

(e) Evaluate $\int_0^1 \int_0^1 \int_0^1 8x^2yz^3 dx dy dz$.

3 (Sem-4/CBCS) MAT HC 1/G 4

3. Answer any four questions : 5×4=20

(a) (i) Find $\frac{\partial w}{\partial r}$ where $w = e^{2x-y+3z^2}$ and

$$x = r + s - t, y = 2r - 3s, z = \cos r t.$$

(ii) Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{x-y}$ does not exist. 2

(b) (i) If f has a relative extremum at $P_0(x_0, y_0)$ and both f_x and f_y exist at (x_0, y_0) , then prove that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0. \quad 2$$

(ii) Discuss the nature of the critical points of the function

$$f(x, y) = (x-2)^2 + (y-3)^4. \quad 3$$

(c) Use a polar double integral to show that a sphere of radius a has volume

$$\frac{4}{3} \pi a^3.$$

3 (Sem-4/CBCS) MAT HC 1/G 5 Contd.

2. Answer the following questions :

2×5=10

(a) Show that the function f is continuous at $(0, 0)$ where

$$f(x, y) = \begin{cases} \frac{y \sin \frac{1}{x}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(b) Compute the slope of the tangent line to the graph of $f(x, y) = x^2 \sin(x + y)$ at the point $P\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ in the direction parallel to the XZ plane.

(c) Evaluate $\iint_R \frac{2xy}{x^2+1} dA$ where $0 \leq x \leq 1$, $1 \leq y \leq 3$.

(d) Show that $\text{div } F = 0$ and $\text{curl } F = 0$, if F is a constant vector field.

(e) Evaluate $\int_0^1 \int_0^1 \int_0^1 8x^2yz^3 dx dy dz$.

3 (Sem-4/CBCS) MAT HC 1/G 4

(d) An object moves in the force field $F = y^2i + 2(x+1)yj$. How much work is performed as the object moves from the point $(2, 0)$ counterclockwise along the elliptical path $x^2 + 4y^2 = 4$ to $(0, 1)$, then back to $(2, 0)$ along the line segment joining the two points.

(e) (i) Evaluate $\int_C (x^2 + y^2) dx + 2xy dy$ where C is the quarter circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$. 3

(ii) A wire has the shape of the curve $x = \sqrt{2} \sin t, y = \cos t, z = \cos t$ for $0 \leq t \leq \pi$. If the wire has density $\delta(x, y, z) = xyz$ at each point (x, y, z) , find its mass. 2

(f) Find the volume of the solid in the first octant that is bounded by the cylinder $x^2 + y^2 = 2y$, the half cone $z = \sqrt{x^2 + y^2}$ and the xy -plane.

3 (Sem-4/CBCS) MAT HC 1/G 6

4. Answer the following questions : 10×4=40

(a) (i) Let $f(x, y)$ be a function that is differentiable at $P_0(x_0, y_0)$. Prove that f has a directional derivative in the direction of the unit vector $u = u_1i + u_2j$ given by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \quad 3$$

(ii) Find the directional derivative of $f(x, y) = \ln(x^2 + y^2)$ at $P_0(1, -3)$ in the direction of $v = 2i - 3j$ using the gradient formula. 3

(iii) Find the equations of the tangent plane and the normal line to the cone $z^2 = x^2 + y^2$ at the point where $x = 3, y = 4$ and $z > 0$. 4

OR

(i) Prove that if $f(x, y)$ is differentiable at (x_0, y_0) , then it is continuous there. 4

3 (Sem-4/CBCS) MAT HC 1/G 7 Contd.

(iii) When two resistances R_1 and R_2 are connected in parallel, the total resistance R satisfies $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

If R_1 is measured as 300 ohms with a maximum error of 2% and R_2 is measured as 500 ohms with a maximum error of 3%, then find the maximum percentage error in R . 6

(b) (i) Use the method of Lagrange multipliers to minimize

$$f(x, y) = x^2 - xy + 2y^2 \text{ subject to } 2x + y = 22. \quad 5$$

(ii) Find all relative extrema and saddle points on the graph of $f(x, y) = x^2y^4$. 5

OR

(i) Find the absolute extrema of the function $f(x, y) = e^{x^2 - y^2}$ over the disk $x^2 + y^2 \leq 1$. 6

(ii) Suppose E be an extreme value of f subject to the constraint $g(x, y) = C$. Prove that the Lagrange multiplier λ is the rate of change of E with respect to C . 4

(c) (i) Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2} dy dx$ by converting to polar coordinates. 5

(ii) Evaluate $\iiint_D e^z dv$ where D is the region described by the inequalities $0 \leq x \leq 1$, $0 \leq y \leq x$ and $0 \leq z \leq x+y$. 5

OR

(i) Find the volume of the solid bounded above by the plane $z = y$ and below in the xy -plane by the part of the disk $x^2 + y^2 \leq 1$ in the 1st quadrant. 5

Contd.

(ii) Evaluate $\iiint_D x dV$ where D is the solid in the 1st octant bounded by the cylinder $x^2 + y^2 = 4$ and the plane $2y + z = 4$. 5

(d) (i) Let C be a piecewise smooth curve that is parameterized by a vector function $R(t)$ for $a \leq t \leq b$ and let F be a vector field that is continuous on C . If f is a scalar function such that $F = \nabla f$, then prove that $\int_C F \cdot dR = f(Q) - f(P)$ where $Q = R(b)$ and $P = R(a)$ are the end points of C . Using it evaluate the line integral $\int_C F \cdot dR$, where $F = \nabla(e^x \sin y - xy - 2y)$ and C is the path described by $R(t) = \left[t^3 \sin \frac{\pi}{2} t \ i - \left[\frac{\pi}{2} \cos \left(\frac{\pi}{2} t + \frac{\pi}{2} \right) \right] j \right]$ for $0 \leq t \leq 1$ 5+3=8

(ii) Determine whether the vector field $F(x, y) = \frac{(y+1)i - xj}{(y+1)^2}$ is conservative. 2

OR

(i) Evaluate $\oint_C \left(\frac{1}{2} y^2 dx + z dy + x dz \right)$ where C is the curve of intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented counterclockwise as viewed from above. 6

(ii) Evaluate $\iint_S F \cdot NdS$ where $F = x^2i + xyj + x^3y^3k$ and S is the surface of the tetrahedron bounded by the plane $x + y + z = 1$ and the coordinate planes with outward unit normal vector N . 4

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