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3 (Sem - 2 / CBCS) MAT HC 1

2024

MATHEMATICS

(Honours Core)

Paper : MAT-HC-2016

(Real Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed :  $1 \times 10 = 10$ 
  - (a) Define,  $\varepsilon$ -neighbourhood of a point  $a$  in  $\mathbb{R}$ , where  $\varepsilon > 0$ .
  - (b) Give an example of a set which is not bounded above.
  - (c) Write the Archimedean property of  $\mathbb{R}$ .

(d) What is the limit of the sequence  $\{x_n\}$ ?

where  $x_n = \frac{2n}{n^2 + 1}$ ?

(e) "The unit interval  $[0, 1]$  is uncountable."

(State True or False)

(f) For what value of  $p$ , the  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges?

(g) "A convergent sequence of real numbers is a Cauchy sequence."

(State True or False)

(h) The sequence  $\{0, 2, 0, 2, 0, 2, \dots\}$  converges to the number 0.

(State True or False)

(i) Give an example of a Cauchy sequence in  $\mathbb{R}$ .

(j) Let  $\{x_n\}$  be a non-zero sequence of real numbers such that  $r = \lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right|$ . Then

$\sum x_n$  is absolutely convergent if

- (i)  $r < 1$

(ii)  $0 \leq r \leq 1$

(iii)  $r > 1$

(iv)  $1 \leq r \leq 2$

(Choose the correct option)

2. Answer the following questions :  $2 \times 5 = 10$

(a) Determine the set  $A$  of  $x \in \mathbb{R}$  such that  $|2x + 3| < 7$ .

(b) Define supremum of a non-empty subset  $S$  of  $\mathbb{R}$ . Write the supremum of the set  $S = \{x \in \mathbb{R} : 0 < x < 1\}$ .

(c) Show that the sequence  $\left\{ \frac{2n-7}{3n+2} \right\}$  is bounded.

(d) Show that the series  $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$  is convergent.

(e) State Cauchy's root test for convergence.

3. Answer **any four** questions :  $5 \times 4 = 20$

(a) If  $x > -1$ , then prove that  $(1+x)^n \geq 1+nx$   
 $\forall n \in \mathbb{N}$ .

(b) If  $x$  and  $y$  are any real numbers with  $x < y$ , then show that there exists a rational number  $r \in \mathbb{Q}$  such that  $x < r < y$ .

(c) Prove that a convergent sequence of real numbers is bounded.

(d) Show that  $\lim_{n \rightarrow \infty} \left( n^{\frac{1}{n}} \right) = 1$

(e) Use comparison test to test convergence of the series whose  $n^{\text{th}}$  term is  $\frac{1}{\sqrt{n+1}}$ .

(f) Show that every absolutely convergent series is convergent. Is the converse true? Justify.  $4+1=5$

4. Answer the following questions :  $10 \times 4 = 40$

(a) For  $a, b \in \mathbb{R}$ , prove that—

(i)  $|a+b| \leq |a|+|b|$

(ii)  $||a|-|b|| \leq |a-b|$

(iii)  $|a-b| \leq |a|+|b|$   $5+3+2=10$

**Or**

Prove that there exists a positive real number  $x$  such that  $x^2 = 2$ .

(b) (i) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound.  $5$

(ii) Show that a sequence in  $\mathbb{R}$  can have at most one limit.  $5$

**Or**

(i) Show that a bounded sequence of real numbers has a convergent subsequence.  $5$



(ii) State and prove that nested interval theorem. 5

(c) Suppose that  $X = (x_n)$ ,  $Y = (y_n)$  and  $Z = (z_n)$  are sequences of real numbers such that  $x_n \leq y_n \leq z_n$  for all  $n \in \mathbb{N}$  and that  $\lim x_n = \lim z_n$ . Then prove that

$Y = (y_n)$  is convergent and  $\lim(x_n) = \lim(y_n) = \lim(z_n)$ . Use the result

to show that  $\lim_{n \rightarrow \infty} \left( \frac{\sin n}{n} \right) = 0$ .

$$7+3=10$$

Or

Show that the sequence  $\left\{ \left( 1 + \frac{1}{n} \right)^n \right\}, n \in \mathbb{N}$

is convergent and  $\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e$  where

$$2 < e < 3.$$

(d) (i) Define absolute convergence of a series in  $\mathbb{R}$ . 2

(ii) Let  $\sum u_n$  be any absolutely convergent series in  $\mathbb{R}$ . Then show that any rearrangement  $\sum v_n$  of  $\sum u_n$  converges to the same value. 7

(iii) Write one rearrangement of the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$ . 1

Or

Discuss convergence of the geometric series  $\sum_{n=0}^{\infty} r^n$ , where  $r \in \mathbb{R}$ .

