

(b) Prove that the only ideals of a field are  $\{0\}$  and  $F$  itself.

(c) Show that the ring of integers is an Euclidean domain.

(d) If  $R$  is a commutative ring with unity and  $A$  is an ideal of  $R$ , show that  $R/A$  is a commutative ring with unity.

(e) Let  $f(x) = x^3 + 2x + 4$  and  $g(x) = 3x + 2$  in  $Z_5[x]$ . Determine the quotient and remainder upon dividing  $f(x)$  by  $g(x)$ .

3. Answer **any four** questions of the following:  
5×4=20

(a) Prove that

$$Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Z\}$$

is a ring under the ordinary addition and multiplication of real numbers.

(b) (i) If  $I$  is an ideal of a ring  $R$  such that  $1$  belongs to  $I$ , then show that  $I = R$ .

(ii) Let  $R$  be a ring and  $a \in R$ . Show that  $S = \{r \in R \mid ra = 0\}$  is an ideal of  $R$ .  
2+3=5

(c) Prove that the ring of integers  $Z$  is a principal ideal domain.

(d) Let  $\phi$  be a homomorphism from a ring  $R$  to a ring  $S$ . If  $A$  is a subring of  $R$  and  $B$  is an ideal of  $S$ , prove that

(i)  $\phi(A) = \{\phi(a) \mid a \in A\}$  is a subring of  $S$ .

(ii)  $\phi^{-1}(B) = \{x \in R \mid \phi(x) \in B\}$  is an ideal of  $R$ .  
2½+2½=5

(e) Let  $F$  be a field,  $a \in F$  and  $f(x) \in F[x]$ . Prove that  $a$  is a zero of  $f(x)$  if and only if  $x - a$  is a factor of  $f(x)$ .

(f) Let  $F$  be a field,  $I$  a nonzero ideal in  $F[x]$ , and  $g(x)$  an element of  $F(x)$ .

Show that  $I = \langle g(x) \rangle$  if and only if  $g(x)$  is a nonzero polynomial of minimum degree in  $I$ .

Answer **either** (a) and (b) **or** (c) and (d) of the following questions :  
10×4=40

4. (a) Prove that a finite integral domain is a field. Hence show that for every prime  $p$ ,  $Z_p$ , the ring of integers modulo  $p$ , is a field.  
4+2=6

(b) Show that  $\frac{R[x]}{\langle x^2 + 1 \rangle}$  is a field. 4

**OR**

(c) Prove that every field is an integral domain. Is the converse true? Justify with an example. 2+1=3

(d) Define prime ideal and maximal ideal of a ring. Show that  $\langle x \rangle$  is a prime ideal of  $Z[x]$  but not a maximal ideal of it. 2+5=7

5. (a) Let  $\phi$  be a homomorphism from a ring  $R$  to a ring  $S$ . Prove that  $\phi$  is an isomorphism if and only if  $\phi$  is onto and  $\ker \phi = \{r \in R \mid \phi(r) = 0\} = \{0\}$ . 5

(b) If  $\phi$  is an isomorphism from a ring  $R$  to a ring  $S$ , then show that  $\phi^{-1}$  is an isomorphism from  $S$  to  $R$ . 5

**OR**

(c) Let  $R$  be a ring with unity  $e$ . Show that the mapping  $\phi : \mathbb{Z} \rightarrow R$  given by  $n \rightarrow ne$  is a ring homomorphism. 5

(d) Define kernel of a ring homomorphism. Let  $\phi$  be a homomorphism from a ring  $R$  to a ring  $S$ . Prove that  $\ker \phi$  is an ideal at  $R$ . 1+4=5

6. (a) State and prove the second isomorphism theorem for rings. 1+7=8

(b) Let  $R$  be a commutative ring of characteristic 2. Show that the mapping  $a \rightarrow a^2$  is a ring homomorphism from  $R$  to  $R$ . 2

**OR**

(c) State and prove the third isomorphism theorem for rings. 1+6=7

(d) Prove that every ideal of a ring  $R$  is the kernel of a ring homomorphism of  $R$ . 3

7. (a) Let  $F$  be a field. If  $f(x) \in F[x]$  and  $\deg f(x) = 2$  or 3, then prove that  $f(x)$  is reducible over  $F$  if and only if  $f(x)$  has a zero in  $F$ . 4

(b) In a principal ideal domain prove that an element is an irreducible if and only if it is a prime. 6

OR

(c) Let  $p$  be a prime and suppose that  $f(x) \in Z[x]$  with  $\deg f(x) \geq 1$ . Let  $\overline{f(x)}$  be the polynomial in  $Z_p[x]$  obtained from  $f(x)$  by reducing all the coefficients of  $f(x)$  modulo  $p$ . If  $f(x)$  is irreducible over  $Z_p$  and  $\deg \overline{f(x)} = \deg f(x)$ , then prove that  $f(x)$  is irreducible over  $Q$ . 5

(d) Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \in Z[x].$$

If there is a prime  $p$  such that

$$p \nmid a_n, p \mid a_{n-1}, \dots, p \mid a_0 \text{ and } p^2 \nmid a_0,$$

then prove that  $f(x)$  is irreducible over  $Q$ . 5

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3 (Sem -4/CBCS) MAT HC3

2023

## MATHEMATICS

(Honours Core)

Paper : MAT-HC-4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions :  $1 \times 10 = 10$ 
  - (a) Give an example of an infinite noncommutative ring that does not have a unity.
  - (b) Define an integral domain.
  - (c) What is the characteristic of the ring of  $2 \times 2$  matrices over integers?
  - (d) In an integral domain, if  $a \neq 0$  and  $ab = ac$ , then prove that  $b = c$ .

- (e) Show that  $2Z \cup 3Z$  is not a subring of  $Z$ .
- (f) Prove that the correspondence  $x \rightarrow 5x$  from  $Z_5$  to  $Z_{10}$  does not preserve addition.
- (g) Characteristic of every field is
- 0
  - an integer
  - either 0 or prime
  - either 0 or not prime
- (Choose the correct option)
- (h) Which of the following is not an integral domain ?
- $Z[x]$
  - $\{a + b\sqrt{2} : a, b \in Z\}$
  - $Z_3$
  - $Z_6$
- (Choose the correct option)

- (i) Consider  $f(x) = 2x^3 + x^2 + 2x + 2$  and  $g(x) = 2x^2 + 2x + 1$  in  $Z_3[x]$ . Then  $f(x) + g(x)$  is
- $2x^3 + x$
  - $2x^2 + 3x + 3$
  - $x^5 + 2$
  - $x^5 + 2x^3 + 2$
- (Choose the correct option)
- (j) The polynomial  $f(x) = 2x^2 + 4$  is irreducible over
- $Q$
  - $C$
  - $Z$
  - None of the above
- (Choose the correct option)

2. Answer the following questions :  $2 \times 5 = 10$
- (a) Let  $R$  be a ring. Prove that  $a(-b) = (-a)b = -(ab)$ , for all  $a, b \in R$ .