Chapter 1

PREREQUISITES

1. PREREQUISITES

In this chapter we present all necessary preliminaries which we require for our work. The concepts discussed in this chapter are available in the standard literature. This chapter contains five sections. In the first section some basic definitions and results of near-rings, near-ring groups, their substructures and homomorphisms and some well known lemmas are presented. The second section deals with the concepts like direct sums, chain conditions and annihilators of near-rings and near ring groups. The third section illustrates some definitions and results of essential extensions of near-rings and near ring groups. The fourth section contains some definitions and results on injective near ring groups. In the last section some examples of the concepts defined in the previous sections are illustrated.

1.1 NEAR-RINGS, NEAR-RING GROUPS AND HOMOMORPHISMS:

In this section some preliminaries of near-rings, near-ring groups, their substructures and homomorphisms and some well known lemmas are presented.

Near - rings are generalized rings, addition need not be commutative and only one distributive law is postulated.

Definition 1.1.1: A right near - ring is a non-empty set N together with two binary operations '+'' and '.' i.e. a triple (N, +, .) such that

(a) (N , +) is a group (not necessarily abelian)

(b) (N , .) is a semi group

(c) $(n_1 + n_2) . n_3 = n_1.n_3 + n_2.n_3 \quad \forall n_1, n_2, n_3 \in N$ (" right distributive law ") .

For example, let (G, +) be a group and N, the set of transformations of G (G to G itself). We define addition + and multiplication . in N by the rules $(f+g)(v) = f(v) + g(v)$,

$(fg)(v) = f(g(v))$, $f, g \in N$, $v \in G$. Then (N, +, .) is a right near ring.

Definition 1.1.2: Instead of (c) if we take $n_1.(n_2 + n_3) = n_1.n_2 + n_1.n_3 \quad \forall n_1, n_2, n_3 \in N$ (" left distributive law ") then (N, +, .) is called a left near ring.

Every ring is a trivial example of right as well as left near-ring.

For both right near-ring and left near ring the theory goes in a parallel way. Throughout our discussion we will consider only the right near ring since the left distributive law is in some way unnatural in near - rings of functions (from where the basic concept of near-ring arises).

We abbreviate (N , + , .) by N. Multiplication of two elements will be indicated by juxtaposition; so we write n_1n_2 instead of $n_1.n_2$.

Definitions 1.1.3: For a near-ring N, $0n = 0 \ \forall n \in N$, where 0 is the identity of the additive group N. This element 0 is called the zero of N.

Since the near-ring N need not satisfy left distributive law there may exist $n \in N$ such that $n0 \neq 0$. $N_0 = \{ n \in N \,/\, n0 = 0 \}$ is called the zero - symmetric part of N. If $N = N_0$, N is called a zero symmetric near-ring.

Proposition 1.1.4: If n_1, n_2 are any elements of a near-ring N, $(-n_1)\, n_2 = -(n_1n_2)$

Definition 1.1.5: If the semi group (N, .) of the near-ring N possesses an identity it is the identity of the near-ring N. And N is called a near-ring with identity or unity.

Unless otherwise mentioned we restrict N for zero symmetric near-ring with unity.

Definition 1.1.6: An element d $\in$ N is called distributive if $\forall n_1, n_2 \in N$, $d(n_1 + n_2) = dn_1 + dn_2$.

Proposition 1.1.7: If d is any distributive element of N, n is an arbitrary element of N, then

$d0 = 0,\ d(-n) = -(dn),\ (-d)(-n) = dn$.

Definition 1.1.8: Let $N_d = \{ d \in N \,/\, d \text{ is distributive} \}$ be a multiplicative subsemi-group of N.

If (N, +) is generated by N_d, N is called a distributively generated near-ring (dgnr).

If N is a dgnr with the generating set N_d then we write $N = \langle N_d \rangle$ and any element $n \in N$ can be written in the form $n = \sum_{\text{finite}} n_{d_i}$, $n_{d_i} \in N_d$.

Definition 1.1.9: Let (E, +) be a group and N a near-ring such that there exists a map $s : N \times E \to E$, $(n, e) \to ne$ satisfying the conditions

i. $(n_1 + n_2)e = n_1e + n_2e$
ii. $(n_1n_2)e = n_1(n_2e)$, for all $n_1, n_2 \in N$, $e \in E$.

Then the triple (E, +,s), simply written E, is a left near-ring group over N or a left N-group.

Throughout our discussion by N-group we mean a left N-group.

Obviously near-ring N is a left N-group.

Every left module over a ring R is a left R-group over the ring R.

The additive group G is an N-group over N, the near-ring of transformations of G.

Definition 1.1.10: An N-group E is called a commutative N-group if (E, +) is an abelian group.

Definitions 1.1.11: The additive identity 0 of the group (E, +) is the zero element of the N-group E.

The N-group E is unitary if there exists an identity $1 \in N$ such that $1e = e$ for every $e \in E$.

Definition 1.1.12: A nonempty subset A of N, where (N, +, .) is a near ring, is a sub-near-ring of N if (A, +, .) is also a near-ring.

Definitions 1.1.13: A non-empty subset A of N is a left N-subgroup of N if A is a subgroup of (N, +) and $na \in A$ for any $a \in A, n \in N$.

A non-empty subset A of N is a right N-subgroup of N if A is a subgroup of (N, +) and

$an \in A$ for any $a \in A, n \in N$.

A subgroup A of (N, +) is a two sided N-subgroup (N-subgroup) of N if $an \in A$ and $na \in A$ for any $a \in A, n \in N$.

Definition 1.1.14: A non-empty subset A of N is a left normal N-subgroup of N if A is a normal subgroup of (N, +) and A is a left N-subgroup of N.

Definitions 1.1.15: A subset A of N is a left ideal of N if A is a normal subgroup of (N, +) and $n_1(a + n_2) - n_1n_2 \in A$ for any $a \in A$, $n_1, n_2 \in N$.

A subset A of N is a right ideal of N if A is normal subgroup of (N, +) and

$an \in A$ for any $a \in A, n \in N$.

A subset A of N is a two sided ideal (ideal) of N if it is a left ideal as well as a right ideal of N.

Proposition 1.1.16: The intersection of any family of N-subgroups (right N-subgroups, left N-subgroups, left ideals, right ideals, ideals) of a near ring is again an N-subgroup (right N-subgroup, left N-subgroup, left ideal, right ideal, ideal) of N.

Proposition 1.1.17: If A is a left ideal and B is a left N-subgroup (left ideal) of near ring N then A + B is a left N-subgroup of N.

Proposition 1.1.18: Every N-subgroup (right N-subgroup, left N-subgroup) of a near-ring N is itself a near-ring.

Proposition 1.1.19: Every ideal of a near-ring N is an N-subgroup of N. And if near-ring N is a dgnr, then every normal(left) N-subgroup is an (left)ideal of N.

Definitions 1.1.20: Let E be an N-group. A subset A of E is an N-subgroup of E if A is a subgroup of the additive group E and $na \in A$ for any $a \in A, n \in N$.

A subset A of E is a normal N-subgroup of E if A is a normal subgroup of the additive group E and $na \in A$ for any $a \in A, n \in N$.

A subset A of E is an ideal of E if A is a normal-subgroup of the additive group E and $n(a + e) - ne \in A$ for any $a \in A, e \in E, n \in N$.

Proposition 1.1.21: The intersection of any family of N-subgroups (ideals) of N-group E is again N-subgroup (ideal) of E.

Proposition 1.1.22: If A is an ideal and B is an N-subgroup (ideal) of N-group E then the A + B is an N-subgroup (ideal) of E.

Proposition 1.1.23: Every N-subgroup of an N-group E is itself an N-group.

Proposition 1.1.24: If A is an N-subgroup of E, B is an N-subgroup of A then B is an N-subgroup of E.

Proposition 1.1.25: If $A \subseteq B$ are N-subgroups of E then A is N-subgroup of B.

Proposition 1.1.26: If A is an N-subgroup of N-group E, I is an N-subgroup of N then for $a \in A$, Ia is an N-subgroup of A.

Proposition 1.1.27: If A, B, C are N-subgroups (ideals) of E with $B \subseteq A$ then $A \cap (B + C) = B + A \cap C$.

Proof: Let $x \in A \cap (B + C)$.

Then $x = a \in A$ and $x = b + c$, for some $b \in B$ and $c \in C$.

So $a = b + c$

$\Rightarrow c = -b + a \in A \cap C$, as $c \in C, -b \in B \subseteq A, a \in A$.

Thus $x = b + c \in B + A \cap C$.

Hence $A \cap (B + C) \subseteq B + A \cap C$ ... ...(i)

Also, if $y \in B + A \cap C$ then $y = b + k$, for some $b \in B$, $k \in A \cap C$.

Now $b \in B \Rightarrow b \in A$.

So, $y = b + k \in A \cap (B + C)$, as $b + k \in A$ and $b + k \in B + C$.

Thus $y \in A \cap (B + C)$.

It gives $B + A \cap C \subseteq A \cap (B + C)$ ... ...(ii)

From (i) and (ii) we get $A \cap (B + C) = B + A \cap C$.

Definitions 1.1.28: A mapping from a near-ring N_1 to a near-ring N_2 i.e. $f : N_1 \rightarrow N_2$ is called a homomorphism if $f(n_1 + n_2) = f(n_1) + f(n_2)$ and $f(n_1n_2) = f(n_1)\, f(n_2)$ for all $n_1, n_2 \in N_1$.

If f is one-one then it is called a monomorphism (N_1 is called embeddable in N_2), if f is onto it is called an epimorphism and if f is both one-one and onto then it is called an isomorphism (N_1 is called isomorphic to N_2 , it is denoted by $N_1 \cong N_2$).

The subset $K = \{ x \in N / f(x) = 0_L\}$ of N is called kernel of f, denoted by Kerf.

Kerf is an ideal of N.

Proposition 1.1.29: For homomorphism $f : N_1 \rightarrow N_2$,if 0 is the zero element of N_1 and if a is any element of N_1,then $f(N_1)$ is a subnear-ring of N_2, $f(0)$ is the zero element of N_2 and $f(-a) = -f(a)$.

Definition 1.1.30: Let E and F be two N-groups. A mapping $f : E \rightarrow F$ is called an N-homomorphism if for all $a, b \in E$, $n \in N$, $f(a + b) = f(a) + f(b)$ and $f(na) = nf(a)$.

The subset $K = \{ x \in E / f(x) = 0_F \}$ of E is called the kernel of N-homomorphism f and is denoted by Kerf. Kerf is an ideal of E.

If f is one-one then it is called an N-monomorphism (E is embeddable in F), if f is onto it is called an N-epimorphism and if f is both one-one and onto then it is called an N-isomorphism (E is called isomorphic to F ,it is denoted by $E \cong F$).

For two N-groups E and F, $Hom_N(E, F)$ denotes the set of all N-homomorphisms { f / f: E → F is N-homomorphism}

Proposition 1.1.31: For N-homomorphism $f : E \to F$, if 0 is the zero element of E and if a is any element of E then f(0) is the zero element of F and $f(-a) = -f(a)$.

Definition 1.1.32: Let N be a near-ring and A an ideal of it. Then the set

$N/A = \{ n + A / n \in N \}$ is a near-ring under the operations addition as

$(a + A) + (b + A) = (a + b) + A$ and multiplication as $(a + A)(b + A) = ab + A$ for $a, b \in N$. This near-ring is called a quotient near-ring.

Definition 1.1.33: Let E be an N-group and A be a normal N-subgroup (ideal) of E. Then the set $E/A = \{ a + A / a \in E \}$ forms an N-group under the operations addition as $(a + A) + (b + A) = (a + b) + A$ and scalar multiplication as $n(a + A) = na + A$ for $a, b \in A$, $n \in N$. It is called quotient N-group.

Proposition 1.1.34: If I is an ideal of near-ring N, then the canonical map $\pi : N \to N/I$, $n \to n + I$ is an epimorphism with kernel I.

Proposition 1.1.35: If N_1, N_2 are two near-rings and $h : N_1 \to N_2$ is an epimorphism, then $N_1/Kerh \cong N_2$.

Proposition 1.1.36: If N is a near-ring and A , B are ideals of N, then

$(A + B)/B \cong A/(A \cap B)$.

Proposition 1.1.37: If N is a near-ring and A, B are normal N-subgroups (ideals) of N such that A contains B, then $(N/B)/(A/B) \cong N/A$.

The corresponding statements for propositions 1.1.34 – 1.1.36 hold for N-group E, replacing near-ring N by E and ideal A of N by A of E.

1.2 DIRECT SUMS , CHAIN CONDITIONS AND ANNIHILATORS:

This section we deals with the concepts like direct sums, chain conditions and annihilators of near-rings and near- ring groups.

Definition 1.2.1: Any family $\{B_i\}$ of N-subgroups of E is said to be an independent family if $B_i \cap (\sum_{i\neq k} B_k) = (0)$, for all distinct $B_0, B_1, B_2, \ldots \ldots \ldots, B_k$.

An ordered set $\{ B_1, B_2, B_3, \ldots \ldots \ldots \}$ of N-subgroups of E is an independent family if $B_i \cap (\sum_{i\neq j} B_j) = (0)$ for all $i \in I^+$.

Definition 1.2.2: Let $(N_i)_{i\in I}$ be a family of near-rings. $\underset{i\in I}{\times} N_i = \{(\ldots, n_i, \ldots) / n_i \in N_i \}$ with component wise defined operations '+' and '.' is called the direct product $\prod_{i\in I} N_i$ of the near-rings N_i $(i \in I)$ is a near-ring.

Definition 1.2.3: Let $(E_i)_{i\in I}$ be a family of N-groups. $\underset{i\in I}{\times} E_i = \{(\ldots, e_i, \ldots) / e_i \in E_i \}$ with componentwise defined operations '+' and '.' as $n(\ldots, e_i, \ldots) = (\ldots, ne_i, \ldots)$ for $n \in N$ is called the direct product $\prod_{i\in I} E_i$ of the N-groups E_i $(i \in I)$ and is an N-group.

The N-subgroup of $\prod_{i\in I} E_i$ consisting of those elements with all components, except a finite number belonging to E_i equal to zero is called the (external) direct sum $\oplus_{i\in I} E_i$. Each E_i is called direct summand of $\oplus_{i\in I} E_i$.

The sub near-ring of $\prod_{i\in I} N_i$ consisting of those elements with all components, except a finite number belonging to N_i equal to zero is called the (external) direct sum $\oplus_{i\in I}N_i$. Each N_i is called direct summand of $\oplus_{i\in I}N_i$.

A sum of finite number of left normal N-subgroups (normal N-subgroups, left ideals, ideals) of N, $A = \sum_{i=1}^{t} A_i$ is called a direct sum of left normal N-subgroups (normal N-subgroups, left ideals, ideals) A_i if each element $a \in A$ can be uniquely expressed in the form $\sum_{i=1}^{t} a_i$, where $a_i \in A_i$. This direct sum is denoted by $A = A_1 \oplus A_2 \oplus \ldots \ldots \ldots \oplus A_t$.

Proposition 1.2.4: If A_i (i = 1,2, ... ,t) are left normal N-subgroups (normal N-subgroups, left ideals, ideals) of N then $A = \sum_{i=1}^{t} A_i$ is a direct sum of left normal N-subgroups (normal N-subgroups, left ideals, ideals) if and only if $A_i \cap (\sum_{i\neq j} A_j) = (0)$.

Note 1.2.5: If $\{ A_i / i \in I \}$ is an independent family of left normal N-subgroups (normal N-subgroups, left ideals, ideals) of N then $\sum_i A_i$ is a direct sum of left normal N-subgroups (normal N-subgroups, left ideals, ideals).

If $\{ A_i / i \in I \}$ is an independent family of normal N-subgroups (ideals) of E then $\sum_i A_i$ is a direct sum of normal N-subgroups (ideals).

Definitions 1.2.6: Let ζ be a non empty collection of subsets of N. A subcollection η of ζ is called a chain if for $A_1, A_2 \in \eta$, either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.

An N-group E is said to be Noetherian or satisfying ascending chain condition or in short acc if every strict ascending chain of N-subgroups $A_1 \subset A_2 \subset \ldots \ldots \ldots$ of E terminates after finitely many steps or equivalently for each chain $A_1 \subseteq A_2 \subseteq \ldots \ldots \ldots$ of E, $\exists\, n \in N$ such that $A_n = A_{n+1} = \ldots \ldots \ldots$.

N-group E is said to be weakly Noetherian if every strict ascending chain of ideals or normal N-subgroups $A_1 \subset A_2 \subset \dots \dots \dots$ of E terminates after finitely many steps or equivalently for each chain $A_1 \subseteq A_2 \subseteq \dots \dots \dots$ of E , $\exists$ n $\in$ N such that $A_n = A_{n+1} = \dots \dots ..$

N-group E is said to be Artinian or satisfying descending chain condition or in short dcc if every descending chain of N-subgroups (ideals or normal N-subgroups) $A_1 \supseteq A_2 \supseteq \dots \dots \dots$ of E, $\exists$ n $\in$ N such that $A_n = A_{n+1} = \dots \dots \dots$.

N-group E is said to be weakly Artinian if every descending chain of ideals or normal N-subgroups $A_1 \supseteq A_2 \supseteq \dots \dots \dots$ of E, $\exists$ n $\in$ N s.t. $A_n = A_{n+1} = \dots \dots \dots$.

Analogously we get ascending chain condition, descending chain condition for substructures in near-ring N also.

Lemma 1.2.7: If A is an N-subgroup (ideal) and B is an ideal of E with $B \subseteq A$ and B, A/B are Noetherian (weakly Noetherian) then A is also Noetherian (weakly Noetherian).

Proof: Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \dots \dots$ be an ascending chain of N-subgroups of A. Since A/B is Noetherian, $\exists$ an integer $m > 0$ such that $A_m + B = A_{m+1} + B$.

Again B is Noetherian and $B \cap A_1 \subseteq B \cap A_2 \subseteq B \cap A_3 \subseteq \dots \dots \dots$ be an ascending chain of N-subgroups of B, so $\exists$ an integer $n > 0$, such that $B \cap A_n = B \cap A_{n+1}$.

Now for $t = \max(m, n)$, $A_t = A_t \cap (A_t + B)$

$$= A_t \cap (A_{t+1} + B)$$

$$= A_{t+1} + (A_t \cap B) \text{ [by proposition 1.1.27]}$$

$$= A_{t+1} + (A_{t+1} \cap B)$$

$$= A_{t+1}$$

Thus A is Noetherian.

If A is an N-subgroup (ideal) of E and A is essential in E (when B = E) then we say that A is an essential N-subgroup (ideal) of E.

Definition 1.3.2: If A, B are two N-subgroups of E such that $A \subseteq B$ then A is weakly essential in B when any non-zero ideal C of E contained in B has a nonzero intersection with A. It is denoted by $A \leq_{we} B$.

Proposition 1.3.3: If A, B, C are N- subgroups E, with $A \subseteq B \subseteq C$, then $A \leq_e C$, if and only if $A \leq_e B \leq_e C$.

Proof: First consider $A \leq_e B \leq_e C$.

Let M be a non zero N- subgroup o f E such that $M \subseteq C$.

Since $B \leq_e C$, $M \cap B \neq 0$ and since $A \leq_e B$, we get $(M \cap B) \cap A \neq 0$ i.e. $M \cap A \neq 0$.

Thus $A \leq_e C$.

Conversely, suppose $A \leq_e C$. Since any non zero N subgroup of C has non zero intersection with A, so $A \cap B \neq 0$ as $B \subseteq C$.

If K is a non zero N-subgroup of E such that $K \subseteq B \subseteq C$, then K is a non zero N-subgroup of C and as $A \leq_e C$, $K \cap A \neq 0$. This gives $A \leq_e B$.

Again if H is any non zero N- subgroup of E with $H \subseteq C$, then $A \cap H \neq 0$ as $A \leq_e C$. So $B \cap H \neq 0$ as $A \subseteq B$ which gives $B \leq_e C$.

Thus $A \leq_e C \Rightarrow A \leq_e B \leq_e C$.

Since $M \cap f^{1}(A) = 0$, we obtain $f(M) \cap A = 0$, which is impossible.

Hence $B_1 \cap B_2 = 0$. So $\{ B_1, B_2\}$ independent.

Applying proposition 1.3.5. to the projection maps

$f_1: B_1 \oplus B_2 \rightarrow B_1$

$f_2: B_1 \oplus B_2 \rightarrow B_2$

$\because A_1 \leq _e B_1$

$\therefore f^{-1}(A_1) = A_1 \oplus B_2 \leq_e B_1 \oplus B_2,$

similarly

$f^{-1}(A_2) = B_1 \oplus A_2 \leq_e B_1 \oplus B_2$

So by preposition1.3.4.,

$(A_1 \oplus B_2) \cap B_1 \oplus A_2 \leq_e B_1 \oplus B_2$

Let $x \in A_1 \oplus A_2$

$\therefore x = a_1 + a_2 \in A_1 \oplus B_2$ as $A_2 \subseteq B_2$

& $x = a_1 + a_2 \in B_1 \oplus A_2$ as $A_1 \subseteq B_1$

$\therefore x \in (A_1 \oplus B_2) \cap (B_1 \oplus A_2)$

Next let $x \in (A_1 \oplus B_2) \cap (B_1 \oplus A_2)$

$\therefore x = a_1 + b_2, x = b_1 + a_2$

$\Rightarrow a_1 + b_2 = b_1 + a_2$

$\Rightarrow\ -b_1 + a_1 = -b_2 + a_2 \in B_1 \cap B_2$

$\Rightarrow a_1 = b_1,\ a_2 = b_2$

i.e. $x = a_1 + b_2 = b_1 + a_2$ i.e. x is uniquely expressible.

i.e. $x \in A_1 \oplus A_2$

$\therefore A_1 \oplus A_2 = (A_1 \oplus B_2) \cap (B_1 \oplus A_2)$

i.e. $A_1 \oplus A_2 \leq_e B_1 \oplus B_2$

Thus the result holds for two elements.

Let the result holds for sets with n −1 elements. Then $\{ B_1, B_2, \ldots \ldots \ldots, B_{n-1}\}$ independent and $\bigoplus_{i=1}^{n-1} A_i \leq_e \bigoplus_{i=1}^{n-1} B_i$

By using the above case $(\bigoplus_{i=1}^{n-1} B_i) \cap B_n = 0$, whence $\{ B_1, B_2, \ldots, B_n\}$ independent and

$(A_1 \oplus A_2 \oplus \ldots \oplus A_{n-1}) \oplus A_n \leq_e (B_1 \oplus B_2 \oplus \ldots \oplus B_{n-1}) \oplus B_n$

Thus the result holds for finite sets.

Corollary 1.3.7: If $\{A_\alpha\}$ is an independent family of normal N subgroups (ideals) of E such that $A_\alpha \leq_e B_\alpha \leq E$ for each α, where B_α's are normal N-subgroups (ideals) of the N-group E, then $\{B_\alpha\}$ is an independent family and $\oplus A_\alpha \leq_e \oplus B_\alpha$.

Proof: For distinct indices $\alpha(0), \alpha(1), \ldots, \alpha(n)$, we know that $\{B_{\alpha(0)}, B_{\alpha(1)}, \ldots, B_{\alpha(n)}\}$ is independent, whence $B_{\alpha(0)} \cap (B_{\alpha(1)} + \ldots + B_{\alpha(n)}) = 0$. Thus $\{B_\alpha\}$ is independent family.

Now any non-zero N-subgroup M of $\oplus B_\alpha$ contains a non-zero element, which must belong to $B_{\alpha(1)} + \ldots + B_{\alpha(n)}$ for some $\alpha(i)$. As a result $M \cap (B_{\alpha(1)} + \ldots + B_{\alpha(n)}) \neq 0$, from which we obtain $M \cap (B_{\alpha(1)} + \ldots + B_{\alpha(n)}) \cap (A_{\alpha(1)} + \ldots + A_{\alpha(n)}) \neq 0$.

And consequently $M \cap (\oplus A_\alpha) \neq 0$.

Definition 1.3.8: Let A be an N-subgroup of C. A complement for A in C is any N-subgroup B of C which is maximal with respect to the property $A \cap B = 0$.

Definition 1.3.9: Let A be an N-subgroup of C. A weak complement for A in C is any ideal B of C which is maximal with respect to the property $A \cap B = 0$.

Proposition 1.3.10: Let A be an ideal (normal N-subgroup) of C. If B is any complement for A in C, then $A \oplus B \leq_e C$.

Proof: Since $A \cap B = 0$, we have $A + B = A \oplus B$, so that $A \oplus B$ is an ideal of C.

Suppose that M is an ideal of C such that $(A \oplus B) \cap M = 0$.

Then the sum $(A \oplus B) + M$ is direct, i.e. $(A \oplus B) + M = (A \oplus B) \oplus M$,

whence $A \cap (B \oplus M) = 0$.

By the maximility of B, we obtain $B \oplus M = B$.

Thus $M = 0$.

Therefore $A \oplus B \leq_e C$.

Definition 1.3.11: An N-group E is called a simple N-group if it has no proper ideals.

Proposition 1.3.12[G. Mason]: The following are equivalent for N-group E

(a) Every ideal of E is a direct summand.

(b) E is a sum of simple ideals.

(c) E is direct sum of simple ideals.

Definitions 1.3.13: The socle of N-group E, denoted by Soc(E), is defined as

Soc(E) = sum of simple ideals.

Equivalently,

Soc(E) = direct sum of simple ideals.

An N-group E is called semisimple if Soc(E) = E that is if one of the conditions of the above proposition1.3.12 is satisfied.

The near-ring N is semi simple if $_NN$ is a semi simple N-group .

Proposition 1.3.14 [Saikia and Choudhury]: If A is any N-subgroup of N-group B, then

Soc(A) = A $\cap$ Soc(B).

Preposition 1.3.15: Let A be an ideal of N- group C. Then A is intersection of essential ideals of C if and only if soc(C) $\leq$ A. In particular soc(C) is the intersection of all the essential ideals of C.

Proof: If S is a simple ideal of C and B is an essential ideal of C, then S $\cap$ B $\neq$ 0 from which we get S $\cap$ B = S i.e. S $\subset$ B.

Thus every simple ideal of C is contained in every essential ideal of C.

If A is intersection of essential ideals of C, this shows that soc(C) ≤ A.

Conversely assume that soc(C) ≤ A and let K denote the intersection of all those essential ideals of C which contain A.

We first claim that any J ≤ K which contains A must be a direct summand of K.

According to proposition 1.3.10 we have J ⊕ B $\leq_e$ C for a suitable B and since A ≤ J ⊕ B, we must have K ≤ J ⊕ B.

Now J ≤ K ≤ J ⊕ B, hence J must be a direct summand of K.

In particular, taking J = A, we obtain K = A ⊕ T for some T.

Given any M ≤ T, we have A ≤ A ⊕ M ≤ K and so A ⊕ M must be a direct summand of K.

Then we have M ≤ T ≤ K with M a direct summand of K, from which it follows that M is a direct summand of T.

Thus every ideal of T is a direct summand of T.

According to proposition 1.3.12, T must be semi-simple, whence

T = Soc(T) ≤ Soc (C)

[since T ≤ C, by proposition 1.3.14].

As Soc(C) ≤ A, we obtain T = T ∩ A = 0 and so K = A.

Therefore A is an intersection of essential ideals of C.

1.4 INJECTIVE NEAR-RING GROUPS:

This section contains some definitions and results on injective near ring groups.

Definition 1.4.1: If A, B, C are N-groups and f, g are N-homomorphisms then a sequence

$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is called a short exact sequence (s.e.s) if f is an N-monomorphism, g is an N-epimorphism and imgf = kerg.

Definition 1.4.2[G. Mason]: If A, B, C are N-groups and f, g are N-homomorphisms then

(a). An N- homomorphism f: $A \rightarrow B$ is normal if f(A) is an ideal of B.

(b). An exact sequence $A \rightarrow B \rightarrow 0$ splits if there exists a normal g : $B \rightarrow A$ such that $fg = 1_B$.

(c). The short exact sequence (s.e.s) $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ splits if the sequence

$B \rightarrow C \rightarrow 0$ splits.

(d). The exact sequence $0 \rightarrow A \rightarrow B$ splits if there exist g : $B \rightarrow A$ such that $gf = 1_A$.

Lemma 1.4.3 [G. Mason]: The following are equivalent

(a). The short exact sequence (s.e.s) $0 \rightarrow A \xrightarrow{h} B \xrightarrow{f} C \rightarrow 0$ splits.

(b). $B = h(A) \oplus g(C) \cong A \oplus C$ where g is the normal splitting map for f.

(c). The exact sequence $0 \rightarrow A \xrightarrow{h} B$ splits.

where A, B, C are N-groups and h, f, g are N-homomorphisms.

Definition 1.4.4 [G. Mason]: The short exact sequence (s.e.s) $0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$ almost splits if there exists g : $C \rightarrow B$ (not necessarily normal) such that $fg = 1_C$.

Definition 1.4.5 [G. Mason]: An N-group A is loosely injective if every s.e.s.

$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ splits.

Definition 1.4.6: An N-group E is called injective if for the exact row $0 \rightarrow A \xrightarrow{f} B$ and any N-homomorphism $g : A \rightarrow E$, we get some N-homomorphism $h : B \rightarrow E$ such that $g = hf$. Where A, B are any N-groups

. i.e. for every diagram

$$\begin{array}{ccc} 0 \rightarrow A & \xrightarrow{f} & B \\ \downarrow & & \\ E & & \end{array}$$

We get a commutative diagram

$$\begin{array}{ccc} 0 \rightarrow A & \xrightarrow{f} & B \\ g \downarrow & \swarrow h & \\ E & & \end{array}$$

Definition 1.4.7 [G. Mason]: (a) N-group E is n-injective (for normal injective) if for every diagram

$$\begin{array}{ccc} 0 \rightarrow A & \xrightarrow{g} & B \\ f \downarrow & & \\ E & & \end{array}$$

of N-groups and N-homomorphisms in which g is normal, there exists $h : B \rightarrow E$ such that hg = f.

(b) N-group A is called almost injective if every s.e.s. $0 \to A \to B \to C \to 0$ almost splits.

Proposition1.4.8 [G. Mason]: For an N-group E,

(1). E is injective ⇒(2). E is n-injective ⇒ (3) E is loosely injective ⇒ (4). E is almost injective.

Definition 1.4.9: The N-group E is the semi direct sum of its N- subgroups A and B, and write $E = A \dot{\oplus} B$ if A is an ideal and $A \cap B = (0)$. In this case B is called a semi-direct summand of E. Then every $e \in E$ can be expressed uniquely as a + b for some $a \in A$, $b \in B$.

Proposition 1.4.10 [G. Mason]:

a. If $E = A \dot{\oplus} B$, and E is n-injective then so is B.
b. If $E = A \oplus B$ and E is loosely injective, then so are A & B.

Proposition 1.4.11: Let $f : A \to E$ be an N-monomorphism, where E is injective. If $A \leq_e B$, f extends to an N-monomorphism $f' : B \to E$.

Proof: Since E is injective, f must extend to a map $f' : B \to E$.

But $A \cap (\ker f') = \ker f = 0$, so $\ker f' = 0$.

i.e. f' is N-monomorphism.

Definition 1.4.12: N- group E is an injective hull of its N-subgroup (ideal) K if E is injective and $K \subseteq L \subseteq E$, where L is injective N-subgroup (ideal) ⇒ L = E.

Equivalently, E is an injective hull of its N-subgroup (ideal) K if E is injective and E is an essential extension of K.

Preposition 1.4.13: If $A_1, A_2, \ldots \ldots \ldots, A_n$ independent family of normal N- subgroups (ideals) of E, then $E(A_1 \oplus A_2 \oplus \ldots \oplus A_n) = E(A_1) \oplus \ldots \ldots \ldots \oplus E(A_n)$

Proof: Let $B_1 = E(A_1), B_2 = E(A_2) \ldots \ldots \ldots, B_n = E(A_n)$.

Then $B_1, B_2, \ldots \ldots \ldots B_n$ is injective N- groups such that

$A_1 \leq_e B_1 \leq E, A_2 \leq_e B_2 \leq E, \ldots \ldots \ldots, A_n \leq_e B_n \leq E.$

By 1.3.6, $\oplus_{i=1}^{n} A_i \leq_e \oplus_{i=1}^{n} B_i$

Again by theorem 3.3.7, $\oplus_{i=1}^{n} B_i$ is injective as $\forall$ i, B_i injective.

$\Rightarrow \oplus_{i=1}^{n} B_i$ is injective hull of $\oplus_{i=1}^{n} A_i$

$\Rightarrow E(\oplus_{i=1}^{n} A_i) = \oplus_{i=1}^{n} E(A_i)$

Definition 1.4.14: Left N-group E is called quasi-injective if every N-homomorphism of any N-subgroup B of E into E can be extended to an N-homomorphism of E into E.

1.5 SOME EXAMPLES:

In this section some examples of the concepts defined in the previous sections are illustrated.

Example 1.5.1: N-subgroups and ideals of near-ring:

Let $N = Z_6 = \{0, 1, 2, 3, 4, 5\}$ is a set with operations '+' and '.' defined by following tables

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	4	4	0	4	1
2	0	2	2	0	2	2
3	0	0	0	0	0	3
4	0	4	4	0	4	4
5	0	2	2	0	2	5

Then $(Z_6, +, .)$ is a near-ring.

If A = {0, 3}, B = {0, 2, 4} then A, B are left N-subgroups as well as right N-subgroups (i.e N-subgroups) of $N = Z_6$, since $NA \subseteq A$, $AN \subseteq A$ and $NB \subseteq B$, $BN \subseteq B$.

A and B are also ideals of N.

Example 1.5.2: N-group and N-subgroup:

Let E = {0, a, b, c} be the Klein's 4-group which is given by the following table

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and consider the mappings on E defined by the following table

E→E	0	a	b	c
f_0	0	0	0	0
f_1	0	a	b	c
f_2	0	a	b	0
f_3	0	0	b	0
f_4	0	a	0	0
f_5	0	0	b	c
f_6	0	0	0	c
f_7	0	a	0	c
f_8	0	b	0	0
f_9	0	c	0	0
f_{10}	0	c	b	0
f_{11}	0	b	b	0
f_{12}	0	c	b	c
f_{13}	0	b	b	c
f_{14}	0	b	0	c
f_{15}	0	c	0	c

$N = \{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}\}$ is a right near-ring with respect to the following operations:

$(f + g)(x) = f(x) + g(x)$

and $(f.g)(x) = f(g(x))$ for all $f, g \in N$; $x \in E$.

Here E is N-group with respect to the operation

$N \times E \rightarrow E$, $(f_i, e) \rightarrow f_i(e)$.

For N-group E , A = {0, b}, B = {0, c} are N-subgroups, since A, B are subgroups of E and $NA \subseteq A$, $NB \subseteq B$.

Example 1.5.3: Essential N-subgroup:

N = D_8, the dihedral group and multiplication in D_8, defined as following table is a near-ring.

•	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	2a	0	2a
2a	0	0	0	0	0	0	0	0
3a	0	0	0	0	0	2a	0	2a
b	b	b	b	b	b	b	b	b
a+b	b	b	b	b	b	2a+b	b	2a+b
2a+b	b	b	b	b	b	b	b	b
3a+b	b	b	b	b	b	2a+b	b	2a+b

A = {0, b}, B = {0, 2a, b, 2a+b} are N-subgroups of N. A, B are essential N-subgroups of N.

Example 1.5.4: Weakly essential N-subgroup:

N = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,11} is the set with addition and multiplication defined as follows

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	3	2	7	6	5	4	10	11	8	9
2	2	3	0	1	5	4	7	6	11	10	9	8
3	3	2	1	0	6	7	4	5	9	8	11	10
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	4	7	6	11	10	9	8	2	3	0	1
6	6	7	4	5	9	8	11	10	3	2	1	0
7	7	6	5	4	10	11	8	9	1	0	3	2
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	8	11	10	3	2	1	0	6	7	4	5
10	10	11	8	9	1	0	3	2	7	6	5	4
11	11	10	9	8	2	3	0	1	5	4	7	6

.	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	4	4	4	4	8	8	8	8
5	0	0	0	0	4	4	4	4	8	8	8	8
6	0	0	0	0	4	4	4	4	8	8	8	8
7	0	0	0	0	4	4	4	4	8	8	8	8
8	0	0	0	0	8	8	8	8	4	4	4	4
9	0	0	0	0	8	8	8	8	4	4	4	4
10	0	0	0	0	8	8	8	8	4	4	4	4
11	0	0	0	0	8	8	8	8	4	4	4	4

Then N is a near-ring. A = {0, 1}, B = {0, 2}, C = {0, 2, 3}, D = {0, 1, 2, 3}, E = {0, 4, 8} are non-trivial N-subgroups. D = {0, 1, 2, 3} is weakly essential but not essential N-subgroup.

Note 1.5.5: Every weakly essential N-subgroup is not essential but every essential N-subgroup is weakly essential.

Example 1.5.6: Quasi-injective N-group:

N = {0, a, b, c} is the Klein's four group with multiplication

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Then (N, +, .) is a near-ring. Here {0, a}, {0, b}, {0, c} are N-subgroups as well as ideals of N.

f : {0, a} $\rightarrow$ N defined by f(0) = 0, f(a) = a is an N-homomorphism.

f can be extended to f : $_NN \rightarrow {_NN}$ by f(i) = i, $\forall$ i $\in$ N. Similarly we get for N-subgroups {0, b} and {0, c} also. So $_NN$ is quasi-injective.

Since N = {0, a} + {0, b} + {0, c}, where {0, a}, {0, b}, {0, c} are simple ideals of N, so N is semi-simple.

For the remaining chapters by N-subgroup of N we mean left N-subgroup of N unless otherwise mentioned.