INTRODUCTION

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In this chapter a background sketch, objective of the work and outcome of work are briefly presented.

0.1. Background sketch:

A near-ring is an algebraic structure with two binary operations, addition and multiplication satisfying all postulates of ring structure except commutativity of addition and one of the distributive laws. The collection of all maps of a group into itself, under the addition defined pointwise and multiplication defined as composition of maps is a natural example of a near-ring. Dickson, who first contributed towards near-rings via an axiomatic research in 1905. From 1930 onwards the near-ring theory has been properly developed and now it is a well established theory. Now the study and research on near-rings is amplified in a very systematic and continuous way.

The theory of near-ring is well established not only by its substantial theory but for its numerous applications also. For example, in theory of computer science planar near-rings are known to yield excellent balanced incomplete block designs, which in turn produce very efficient designs of statistical experiments. It is widely applied in combinatorial problems, interpolation theory, polynomials and matrices, graph theory.

1930 onwards Fittings, Zassecnhaus, Taussky, H. Wielandt had done extensive work in near-ring theory. Later on Blackett, Frolich, Laxton, Beidleman, Malone, Oswald, Maxon, Meldrum and many others had been engaged in extensive research in different aspects of near-ring theory. Clay, Kandasamy and Pliz have carried out elegant contribution by their books [19], [36] and [51] to near-ring theory. Now there are about half a dozen books on near-rings apart from the conference proceedings.

Blackett [11] studied simple and semi-simple near-rings around 1950. Chowdhury, Mason, Saikia [55] extended these concepts to strictly, strictly-1 semi-simple near-ring groups. Minimal ideals were studied by Blackett and recently Gerhard Wendt, Linz are carrying out work on the study of minimal ideals of near-rings. Ligh Steve [40] studied chain conditions in near -rings. Mason [41], Oswald [48], Yong Uk·CHO [68] studied some concepts of injectivity and projectivity in near-rings.

As near-ring is a generalization of ring theory, it is natural desire to extend the concepts from rings to near-rings. Bhavanari [59], Uma et al [65], Hartney, [31], Ligh Steve, Rao and Prasad [53], Wendt Gerhard [66], Booth & Hall [12] extended various concepts to near-ring theory. For relaxation of commutativity of addition and one of the distributive laws it has a different beauty.

The theory of honest subgroups was developed by Abian and Rinehart in [1]. The concepts of isolated submodules, honest submodules are studied by Fay and Joebert, Jara in [25], [34]. For a skew field, the notions of isolated submodules and honest submodules coincide. The honest submodules lead to a new characterization of ore domain. Moreover following the theory developed by Fay and Joebert, Jara obtained the characterizations of rings of quotients in terms of honest operator. Closure operators have played a fundamental role in a variety of areas of mathematics. In the category of groups isolated subgroups are useful in the study of torsion-free groups. In case of category of modules closure operators like *T*-closed, *T*-honest have been studied by Fay and Joubert.

The concept of super honest submodules was introduced by Joubert and Schoeman [35]. Super honest submodules of quasi injective modules are studied by Cheng [15]. We attempt to extend the notion of super-honesty in modules to near-ring groups. We are also motivated to intricate relation between honesty and superhonesty.

Faith and Utumi [24], Page and Yousif [50], Boyle [13], Goodearl [29], Armendariz [5], Hirano [32], Fuller [27], Fuller and Anderson [3] investigated many properties and relations of quasi- injective, relative injective modules with chain conditions in ring-theory. Injective modules and near-ring groups have been studied by several researchers like Mason et al, Faith et al, Seth and Tiwari [60], Meldrum [46], Oswald [48]. Of these Oswald and Mason have studied injective and projective near-ring modules. Mason [41] studied injective near-ring modules and defined the concepts like n-injective, loosely injective and almost injective near-ring modules. De, Chowdhury and Saikia have shown how the n-injectivity and weakly n-injectivity character together with Beidleman's condition exhibit some interesting phenomena in case of m-simple near-ring groups. Recently, Saikia and Misra[47] have studied p-injective near-rings and weakly quasi injective near-ring groups. This thesis we attempt to extend some characteristics of quasi-injective, relative injective near-rings and near-ring groups.

0.2. Objective of the work:

Our objective is to study near-rings and near-ring groups with injectivity and chain conditions. The study of injective near-rings and near-ring groups is well developed. But there are many scopes for modifications like imposing different conditions on the substructures as well as on the near-ring groups.

Our aim is to extend the notions of honest submodules and super honest submodules to near-rings and near-ring groups and to obtain some structure theorems. We attempt to establish the results that hold for modules and rings.

By generalizing the concept of relative injectivity of modules to near-ring groups we attempt to study relative injective near-ring groups. Imposing conditions like simple, semi-simple and singularity along with chain conditions we try to study various characteristics of relative injective N-groups and try to establish some structure theorems. The above mentioned characters motivate us to derive some interrelated results of relative injective N-groups.

The study of quasi injective modules and their endomorphism rings motivates us to extend these concepts to near-ring groups. The objective of this generalization is to investigate whether analogous results can be obtained in near-ring groups. We attempt to derive the near-ring character of the set of endomorphism of quasi-injective N-groups under certain conditions and this leads us to a near-ring group structure which motivates us to study various characteristics of the structure. We also attempt to study quasi injective N-groups satisfying different chain conditions on their substructures.

0.3. Outcome of the work:

Along with the lines mentioned above the outcome of our work has been elucidated in four chapters.

The **first chapter** of the thesis is "The Prerequisites". In this chapter we present some known definitions and results of near-ring theory which are used in later chapters. This chapter has five sections. In section 1, some basic definitions and results of near-ring, near-ring groups, their substructures and homomorphisms of near-rings, near-ring groups are presented. In the second section the concepts like direct sums, chain condition and annihilators of near-rings and near-ring groups are discussed. The third section deals with essentiality in near-rings and near-ring groups. The fourth section is devoted to the study of injective near-ring groups and related properties. In the last section some of the terms defined in the earlier sections are illustrated with examples.

Chapter 2 is the outcome of our papers [57] which is published in Indian journal of Mathematics and Mathematical Science and [58] which is published in Advances in algebra. In this chapter we attempt to extend the notion of super-honesty and honesty in modules to near-ring groups. This chapter is divided into four sections.

The first section contains the basic required definitions and some preliminary results. In this section considering χ as the set of essential N-subgroups, we define the concepts like χ-honest N-subgroups, χ-closure of N-subgroup, χ-torsion N-subgroup together with superhonest N-subgroups(ideals) and essentially closed N-subgroups. We prove many preliminary results needed for the subsequent sections. In the second section we investigate various characteristics of χ-honest N-subgroups. Using terms like linear filter, topological filter we characterize χ-honest N-subgroups. The property of χ-honest and χ-torsion free of N-subgroups imply the χ-closed character of the N-subgroups and the torsion free character of the N-group. The third section deals with the characterization of superhonest N-subgroups (ideals). Necessary and sufficient conditions for superhonest N-subgroups are established. For an N-subgroup of an N-group E, equivalence of superhonesty, χ-closedness and essentially closedness are established with additional conditions on the χ-torsion part of the subgroup. It is shown that the inverse image of a homomorphism inherits the superhonesty character whereas homomorphic image does not. Using the notion of χ-closed, χ-torsion N-subgroups we establish that under certain conditions every χ-closed N-subgroup of an N-group E is super-honest in E. in the fourth section some special types of χ-honest and superhonest N-groups are studied and related properties are discussed.

The **third chapter** deals with injectivity and relative injectivity of near-ring groups. This chapter is the outcome of the paper "Singularity in E-injective N-group" communicated for publication in International Journal of Pure and applied Mathematics. This chapter is divided into four sections. The first section contains the basic definitions and results. In the second section we define relative injective N-groups, and some special relative injective N-groups and investigate various characters of these N-groups. Several concepts like WI N-groups, V N-groups, SI N-groups, S^2I N-groups, S^3I N-groups are defined. It is proved that the N-subgroups of WI N-groups inherits the WI character whereas under special conditions homomorphic image of WI N- groups are also so. Third section deals with the study of direct sum of E-injective N-groups. Using the notion of dominance of an element of an N-group by another N-group several properties are established. The following results are established.

The direct sum of a family of WI N-groups is also a WI N-group. If W is a commutative N-group and $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of N-group E, W is Ne–injective for all $e \in E$, then W is E- injective . If a finite direct sum of injective normal N-subgroups (ideals) Q_α of E is injective then each Q_α is injective.

E-injective N-groups with chain conditions are studied in the final section. In particular, E-injective N-groups with descending chain condition are investigated. It is shown that the singular and semi-simple characters play a vital role in characterization of E-injective N-groups, For a distributively generated near-ring N if $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of a Noetherian N-group E, $\oplus_{\alpha \in J} A_\alpha$ is commutative N-group then direct sum of any family $\{A_\alpha\}$ of E - injective N- groups is E – injective under certain condition. Moreover in some particular case, in a Noetherian V N-group (V_c N-group) E, every strictly semi- simple N-group is E- injective, whereas weakly

Noetherian V_c N-group character is exhibited in certain case. If E is an N-group satisfying the conditions $\{Ne\}_{e \in E}$ is an independent family of normal N-subgroups of E, direct sum of E-injective N-groups is a commutative N-group, No non-zero homomorphic image of Nx, $\forall x (\neq 0) \in Soc(E)$, is semi-simple, singular and $\frac{E}{Soc(E)}$ is Noetherian V N-group, then E is an strictly semi-simple singular N-group . If E is non-singular and every singular homomorphic image of E is weakly Noetherian then E is almost weakly Noetherian. E is non-singular and almost weakly Noetherian and in E every weakly essential N-subgroup is essential then every singular homomorphic image of E is weakly Noetherian. Several equivalent conditions for N-groups with chain conditions are established.

The **fourth chapter** deals with quasi-injective N-groups and near-rings of endomorphisms of injective hulls of quasi injective N-groups. Some of the contents of this chapter form the paper "Quasi Injective near-ring groups" which is communicated to Far East Journal of Mathematical sciences. This chapter is divided into five sections. In the first section of this chapter we define the basic terms and results that are needed for the sequel. In the second section quasi injective near-ring groups and their endomorphism near-ring are studied. If E is a quasi injective N-group and S = End(injective hull of E) then we study the structure ES and various properties of ES are proved. It is proved that ES is a minimal quasi-injective extension of E and any two minimal quasi-injective extensions are equivalent. This structure motivates to study the Jacobson radical of endomorphism near-ring of quasi-injective N-group E. It is established that the Jacobson radical has the small character and the near-ring modulo the Jacobson radical is a regular near-ring. Third section of this chapter deals with some properties of quasi injective N-groups. Some properties of quasi-injective N-groups relating essentially closed N-subgroups and complement N-subgroups are established in the third section. If E is a finitely generated quasi-injective left N-group then there exists factor group of E with infinite Goldie dimension. In this section quasi injective N-groups satisfying chain conditions are also studied. If a finitely generated quasi-injective left N-group E has ascending chain condition on essential ideals, then E is weakly Noetherian. In the fourth section superhonest N-subgroups of quasi-injective N-groups are studied. If E is a quasi-injective N-group, then closure of the torsion part is the smallest super-honest N-subgroup of E. Any χ-closed ideal of a quasi-injective N-group E is a superhonest ideal in E. If E is a quasi-injective N-group, then the smallest super-honest N-subgroup is the χ-closure of the N-subgroup generated by the torsion part of E with respect to the near-ring N. If E is a quasi-injective N-group with a non trivial super-honest N-subgroup P' then there exists an N-endomorphism f of E such that $f(P') \not\subset P'$. In fifth section we try to establish some relations between quasi injective N-groups and relative injective N-groups.