Chapter 4

QUASI-INJECTIVE N-GROUPS

4. QUASI-INJECTIVE N-GROUPS

This chapter deals with quasi-injective N-groups and near-ring groups.

4.1. PREREQUISITES:

In this section of this chapter we define the basic terms and results that are needed for the sequel.

Definition 4.1.1: For a right near- ring (N, +, .) and a corresponding N- group E, suppose there is an $x \in E$ such that $\{nx / n \in N\} = E$. Then E is a monogenic N - group and x is a generator.

Definition 4.1.2: An N-subgroup B of E is called fully invariant if for each N-homomorphism $f : E \rightarrow E$, $f(B) \subseteq B$.

Definition 4.1.3: A left ideal A of N is called small (strictly small) if N = B for each left ideal (N-subgroup) B such that N = A + B.

Since every left ideal is a left N-subgroup, a strictly small left ideal of N is also a small left ideal of N.

Definition 4.1.4: The intersection of all maximal ideals maximal as N-subgroups of N-group E is called radical of E and is denoted by J(E).

Definition 4.1.5: An N-group E is called irreducible if it has no proper non-zero N-subgroups.

Lemma 4.1.6 [K. Misra]: If the radical ideal J(N) is strictly small in N then the following conditions are equivalent-

(i) $Y \in J(N)$

(ii) $1-xy$ is left invertble for all $x \in N$

(iii) $yM = 0$ for any irreducible left N-group M.

Proposition 4.1.7: Let $0 \to A \to E \to B \to 0$ be a short exact sequence of N-groups where A is N-subgroup (ideal) of E. Then E is Noetherian (weakly Noetherian) if and only if both A and B are Noetherian (weakly Noetherian).

Proof: First let E be Noetherian .

Then since A is isomorphic to an N-subgroup of E, so by definition A is Noetherian .

Again let $g : E \to B$ be the N-epimorphism.

Then $E/\text{Ker}g \cong B$.

Kerg is ideal of E and E is Noetherian, so $E/\text{Ker}g \cong B$ is Noetherian.

Conversely let A and B are both Noetherian, to show E is Noetherian.

If we assume A is an ideal of E and $B = E/A$. Proof of rest part is same as lemma 1.2.7.

If A is an N-subgroup of E, $E/\text{Ker}g \cong B$ is Noetherian.

$\text{Im}f = \text{Ker}g$, Kerg is ideal of E.

Now, A is Noetherian and $A/\text{Ker}f \cong \text{Im}f$

A is Noetherian $\Longrightarrow$ A/Kerf is Noetherian $\Longrightarrow$ Imf is Noetherian $\Longrightarrow$ Kerg is Noetherian .

so E/Kerg, Kerg is Noetherian $\Longrightarrow$ E is Noetherian .

Corollary 4.1.8: If $E = E_1 \oplus E_2 \oplus \ldots \ldots \ldots \oplus E_n$ i.e E is finite direct sum of ideals of N-group E then E is weakly Noetherian if and only if E_1 , E_2, ... , E_n are weakly Noetherian.

In [60] V. Seth and K. Tiwari proved that if N left dgnr , with identity and M right N-group then M is injective if and only if for every right ideal U of N and every N-homomorphism $f : U \longrightarrow M$, there exists an element m in M such that f(a) = ma for all a in U. But in [48] A. Oswald claimed that converse of the above is not always true.

Theorem 4.1.9 [Seth, Tiwari]: N near-ring with identity and M N-group. If M is injective then for for every right ideal U of N and every N-homomorphism $f : U \longrightarrow M$, there exists an element m in M such that f(a) = ma for all a in U.

Theorem 4.1.10: An N-group E is quasi-injective if and only if E is fully invariant N-subgroup of its injective hull.

Proof: Let $S = \mathrm{End}_N \hat{E}$ be the set of N-endomorphisms of $\hat{E}$, $\hat{E}$ injective hull of E,

where $(f + g)e = f(e) + g(e)$ for $f, g \in S$ and $e \in \hat{E}$.

First we assume E is fully invariant N-subgroup of $\hat{E}$. i.e. $fE \subseteq E$, $\forall f \in S$.

Let M be an N-subgroup of E and $t : M \to E$ be an N-homomorphism. Then t must extend to some $f \in S$, so E is quasi-injective.

Next let N-group E be quasi-injective and $f \in S$. To show $fE \subseteq E$.

Restricting f, we get a map $k : E \cap f^{-1}(E) \to E$, i.e. $f(x) = k(x)$ for $x \in E \cap f^{-1}(E)$

where $f^{-1}(E) = \{x \in \hat{E} / f(x) \in E\}$.

Now $E \cap f^{-1}(E)$ is an N-subgroup of E, so by quasi-injectivity of E, k can be extended to an N-endomorphism t of E. i.e. $t(x) = k(x)$ $\forall x \in E \cap f^{-1}(E)$.

Then t extends to a map $g \in S$ such that $g(E) \subseteq E$, so we get

$g : \hat{E} \to \hat{E}$ with $g(x) = t(x), \forall\, x \in E$.

Also $(g - f)(E \cap f^{-1}(E)) = 0$.

For if $x \in E \cap f^{-1}(E)$, then $x \in E$ and $x \in \hat{E}$ such that $f(x) \in E$ and

$(g - f)(x) = g(x) - f(x) = 0$, since for $x \in E \cap f^{-1}(E)$, $g(x) = t(x) = k(x) = f(x)$.

Since $g(E) \subseteq E$ we get $E \cap (g - f)^{-1}E \subseteq (E \cap f^{-1}(E)) \subseteq \ker(g - f)$, where

$(g - f)^{-1}E = \{x \in \hat{E} / (g - f)(x) \in E\}$.

Now, $x \in E \cap (g - f)^{-1}E \Rightarrow x \in E$ and $x \in (g - f)^{-1}E$.

As $x \in E \Rightarrow g(x) \in gE \subseteq E$.

So $f(x) = g(x) - (g(x) - f(x)) \in E$

$\Rightarrow x \in E \cap f^{-1}(E)$

Thus $E \cap (g - f)^{-1}E \subseteq (E \cap f^{-1}(E)) \subseteq \ker(g - f)$ [since $(g - f)(E \cap f^{-1}(E)) = 0$]

$\Rightarrow (g - f)E \cap E = 0$

Since $x \in (g - f)E \cap E \Rightarrow x = (g - f)y$; $x \in E, y \in E$

$\Rightarrow y \in (g - f)^{-1}E$

$\Rightarrow y \in E \cap (g - f)^{-1}E$

$\Rightarrow y \in \ker(g - f)$

$\Rightarrow (g - f)y = 0$.

Now $(g - f)E \cap E = 0 \Rightarrow (g - f)E = 0$, because $E \leq_e \hat{E}$.

Hence $f(E) = g(E) \subseteq E$

$\Rightarrow f(E) \subseteq E$.

Theorem 4.1.11: If E is quasi-injective then its direct summands are also quasi-injective.

Proof: Let the normal N-subgroup A be a direct summand of E. To show A is quasi-injective.

Consider the direct sum decomposition $E = A \oplus B$ for some normal N-subgroup B.

Then by proposition 1.4.13 $\widehat{E} = \widehat{A} \oplus \widehat{B}$ and $S = \text{End}_N \widehat{E}$.

If $p \in S$ is the projection onto $\widehat{A}$, then $pSp = \text{End}_N \widehat{A}$.

Now $SE \subseteq S$ by theorem 4.1.10, whence $pSpE \subseteq pE$ and so $pSpA \subseteq A$.

So again by theorem 4.1.10, A is quasi-injective.

Theorem 4.1.12 [Clay]: For a near-ring (N, +, .) with identity 1, suppose E is a monogenic unitary N-group with generator x and suppose that $T = \{ m \in n \;/\; \text{Ann}(x)m \in \text{Ann}(x)\}$ is a subgroup of (N, +). Then the N-endomorphisms $\text{End}_N E$ of N- group E forms a right near ring where $(f \oplus g)(x) = f(x) + g(x)$ and $(f.g)(x) = f(g(x))$.

Also E is an $\text{End}_N E$ –group defined by

$$\varphi: E \times \text{End}_N E \longrightarrow E \text{ by } \quad \varphi(m.f) = m.f = f(m).$$

4.2 Endomorphism near ring of quasi-injective N-groups:

In this section we investigate various characteristics of endomorphism near-ring of quasi-injective N-groups. We also study some aspects of Jacobson radical of endomorphism near-ring of quasi-injective N- groups.

Throughout this section of this chapter we assume E satisfies the condition of theorem 4.1.12. and N is a dgnr.

If $\hat{E}$- injective hull of E, we consider $S = End_N \hat{E}$

$\phi: \hat{E} \times S \longrightarrow \hat{E}$ by $\phi(m, f) = m.f = f(m), m \in \hat{E}, f \in S$, then $\hat{E}$ is an S-group.

For this S-group we get the following:

Proposition 4.2.1: ES is an N-subgroup of $\hat{E}$.

Let a, b ∈ ES

$a = \sum x_i f_i$, $b = \sum y_j f_j$, $a - b = \sum x_i f_i - \sum y_j f_j \in ES$

Let n ∈ N, a ∈ ES to show na ∈ ES

$a = \sum x_i f_i$

$na = n \sum x_i f_i$

$= n \sum f(x_i)$

$= (s_1 + s_2 + s_3 + \dots \dots \dots + s_n) \sum f_i(x_i)$

$= s_1 \sum f_i(x_i) + s_2 \sum f_i(x_i) + \dots \dots \dots + s_n \sum f_i(x_i)$

$= \sum s_1 f_i(x_i) + \sum s_2 f_i(x_i) + \dots \dots \dots + \sum s_n f_i(x_i)$

$= \sum f_i(s_1 x_i) + \sum f_i(s_2 x_i) + \dots \dots \dots + \sum f_i(s_n x_i)$

$= \sum (s_1 x_i) f_i + \sum (s_2 x_i) f_i + \dots \dots \dots + \sum (s_n x_i) f_i$

$\in ES$ $[\because (s_j x_i) \in E]$

Proposition 4.2.2:

a. ES is quasi- injective
b. ES is the intersection of all quasi-injective N- subgroups of $\widehat{E}$ containing E. So ES is the smallest N-subgroup of $\widehat{E}$ containing E.
c. E is quasi- injective if and only if E = ES .

Proof:

(a) Let M be an N-subgroup of ES & $f : M \longrightarrow ES$ we take the inclusion map $i : ES \longrightarrow \widehat{E}$

Then the composite map $h = if : M \longrightarrow \widehat{E}$.

Since $\widehat{E}$ is injective, so h can be extended by some $\lambda: \widehat{E} \longrightarrow \widehat{E}$ such that

$$
\begin{aligned}
x.\lambda = \lambda(x) &= x.h \text{ for } x \in M \\
&= x.(if) \\
&= (if)(x) \\
&= i(f(x)) \\
&= f(x) \\
&= x.f, \quad \text{where } x.f = f(x) \in ES
\end{aligned}
$$

Thus f is induced by $\lambda \in S$.

Now let $g \in S$. Then for $y = \Sigma\ x_i g_i \in ES$

$(\Sigma\, x_i g_i)\, \lambda = \Sigma\, x_i(g_i\lambda) \in ES \qquad \because g_i\lambda \in S$

$\therefore (ES)\,\lambda \subseteq ES.$

λ induces $\overline{\lambda} : ES \longrightarrow ES$

i.e. λ can be restricted by some $\overline{\lambda}$: ES $\longrightarrow$ ES such that

$x\overline{\lambda} = x.\lambda$ for $x \in ES$

$\therefore x\overline{\lambda} = x.f$ for $x \in M$ ($\because x.\lambda = x.f$ for $x \in M$ and $M \subseteq ES$)

$\Rightarrow$ f is induced by $\overline{\lambda}$: ES $\rightarrow$ ES $\Rightarrow$ES is quasi-injective.

(b) Let P be any quasi-injective N-subgroup of $\widehat{E}$ containing E.

We wish to show ES = ∩P.

Since by (a) ES is quasi-injective. So ∩P ⊆ ES.

Now to show ES ⊆ ∩ P. We will show ES ⊆ P. So it is sufficient to show that $P\alpha \subseteq P$ $\forall\alpha \in S$.

Since if $\forall\ \alpha \in S$, $P\alpha \subseteq P$ then $PS \subseteq P$.

But $E \subseteq P \Rightarrow ES \subseteq PS$ [$\because E \subseteq P \Rightarrow E\lambda \subseteq P\lambda$]

$\Rightarrow ES \subseteq P$.

To prove this we see that

$Q(\alpha) = \{ x \in P / x\alpha \in P \}$ is an N-subgroup of P.

Let $x, y \in Q(\alpha) \Rightarrow x\alpha \in P, y\alpha \in P$.

$x\alpha - y\alpha \in P$.

$\Rightarrow \alpha(x) - \alpha(y) \in P$

$\Rightarrow \alpha(x-y) \in P$

$\Rightarrow x-y \in Q(\alpha)$

Next to show $N\,Q(\alpha) \subseteq Q(\alpha)$

i.e. for $n \in N, x \in Q(\alpha)$ to show $nx \in Q(\alpha)$.

$x \in Q(\alpha) \Rightarrow x \in P$ such that $x.\alpha \in P$

$\because x \in P, n \in N \Rightarrow nx \in P$ ($\because NP \subseteq P$)

$(nx).\alpha = \alpha(nx) = n\alpha(x) = n(x.\alpha) \in P$ ($\because NP \subseteq P$)

$\Rightarrow nx \in Q(\alpha)$.

$\therefore Q(\alpha)$ is an N-subgroup of P.

We have only to show that $Q(\alpha) = P \quad \forall \alpha \in S$, since then $y \in P \Rightarrow y \in Q(\alpha) \Rightarrow y.\alpha \in P \Rightarrow$

$P\alpha \subseteq P$

Since $q \rightarrow q\alpha$, $q \in Q(\alpha) = Q$ a map of Q into P and since P is quasi-injective so there exists

$\alpha_1 : P \rightarrow P$ such that $q\,\alpha_1 = q\,\alpha \quad \forall q \in Q$

Since $\hat{E}$ is injective, $\exists\, \alpha' \in S$ such that $x\,\alpha' = x\,\alpha_1 \quad \forall x \in P$

Since $P\,\alpha' \subseteq P$

If $P(\alpha' - \alpha) = 0$ then $P\alpha' = P\alpha$

So $P\alpha \subseteq P$

So if $Q(\alpha) \neq P$ then $P(\alpha' - \alpha) \neq 0$

As we know $E \leq_e \hat{E} \Rightarrow P \leq_e \hat{E}$

($\because$ if $A(\neq 0) \leq_S \hat{E}$ & $P \cap A = 0$ then $E \cap A = 0$ contradicts $E \leq_e \hat{E}$)

Now $P(\alpha' - \alpha)$ is N-subgroup of $\hat{E}$

a, b $\in P(\alpha'-\alpha)$

$a = p_1(\alpha'-\alpha)$ $b = p_2(\alpha'-\alpha)$

$a-b = p_1(\alpha'-\alpha) - p_2(\alpha'-\alpha)$

$= (p_1- p_2)(\alpha'-\alpha) \in P(\alpha'-\alpha)$ $[\because (\alpha'-\alpha) \in S]$

For $n \in N$, $x \in P(\alpha'-\alpha)$ let $x = p_1(\alpha'-\alpha)$

Now $n\, p_1 (\alpha'-\alpha) = n (\alpha'-\alpha) p_1$

$= n\, \alpha' (p_1) - n\, \alpha(p_1)$

$= \alpha' (n\, p_1) - \alpha(n\, p_1)$

$= (\alpha'-\alpha) (np_1)$

$= (np_1)(\alpha'-\alpha) \in P(\alpha'-\alpha)$

$\therefore P(\alpha'-\alpha)$ N-subgroup of $\hat{E}$.

Consequently we have $P(\alpha'-\alpha) \cap P \neq 0$

But if $x, 0 \neq y \in P$ are such that $y = x (\alpha'-\alpha) \in P(\alpha'-\alpha) \cap P$

Then since $x\, \alpha' = x\, \alpha_1$ $\because x \in P$, $y = x(\alpha'-\alpha) = (\alpha'-\alpha)(x) = (\alpha'x-\alpha x) = x\alpha'-x\alpha$

$\therefore x\, \alpha = x\, \alpha_1 - y \in P$

Then $x \in Q(\alpha)$ so that $x\, \alpha = x\, \alpha'$ and so $y = 0$, a contradiction. Which establishes (b).

(c) Since ES is the intersection of all quasi-injective N-subgroups of $\hat{E}$, containing E.

E is quasi-injective $\Longrightarrow$ ES $\subseteq$ E. And E $\subseteq$ ES is obvious by inclusion map.

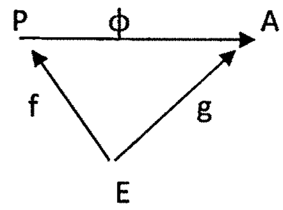
$\therefore$ ES = E

Definition 4.2.3: (P, E, f) denotes a N-monomorphism $f : E \rightarrow P$ and is called an extension of E.

An extension (P, E, f) of an N-group E is a minimal quasi-injective extension in case P is quasi-injective and the following condition is satisfied:

If (A, E, g) is any quasi-injective extension of E, then there exists an N-monomorphism

$\phi : P \rightarrow A$ such that



commutes i.e. $g = \phi f$.

Proposition 4.2.4: ES is minimal quasi-injective extension of E. Any two minimal quasi-injective extensions are equivalent.

Proof: Let (A, E, g) be any quasi-injective extension of E.

Let $\widehat{A} = E(A)$ & $\Omega = Hom_N(\widehat{A}, \widehat{A})$

Then by proposition 4.2.2 $A\Omega \subseteq A$.

Since ES is an essential extension of E, the N-monomorphism $g : E \rightarrow \widehat{A}$ can be extended to a monomorphism (also denoted by g) of ES in $\widehat{A}$.

[$\because$ if $f : A \xrightarrow{mono} E$, E injective , $A \leq_e B$, then f extends to $f' : B \xrightarrow{mono} E$]

Since g(ES) is quasi-injective .

[$\because$ $g(ES) \cong ES$, $\because$ Kerg = 0 ($f : A \xrightarrow{mono} B$, $A \cong f(A)$)]

Then (g(ES)) $\Omega \subseteq$ g(ES) and we conclude that (B) $\Omega \subseteq$B where B = g(ES) $\cap$ A $\subseteq$ g(ES)

g^{-1} (B) $\subseteq$ (ES)

[$\because$ AB $\subseteq$ B, AC $\subseteq$ C, A(B $\cap$ C) = AB $\cap$ AC $\subseteq$ B $\cap$ C]

Since B $\subseteq$ (B)Ω is obvious.

$\therefore$ by proposition 4.2.2. B is quasi-injective.

It follows that g^{-1} (B) is a quasi-injective extension of E $\subseteq$ES.

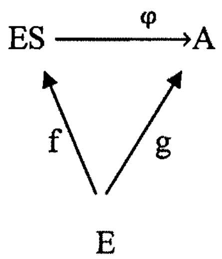
Since ES is the smallest quasi-injective extension of E contained in $\hat{E}$, we conclude that

g^{-1}(B) = ES. So B = g(ES) $\subseteq$ A. This establishes that ES is a minimal quasi-injective extension.

Next if (A, E, g) is also a minimal quasi-injective extension of E, then (A, E, g) is also equivalent to ES.

ES minimal quasi-injective extension of E. (A, E, g) also quasi-injective extension of E.

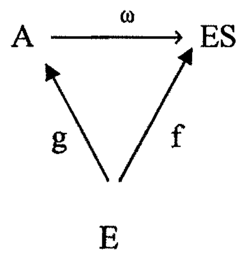
By definition for $E \xrightarrow{\text{mono } \varphi} ES$, $E \xrightarrow{\text{mono } f} A$, there exists $ES \xrightarrow{\text{mono } \varphi} A$ s.t. the diagram



commutes i.e. g = φf.

Again (A, E, g) is minimal quasi-injective extension of E. ES is also quasi-injective extension of E. By definition for $E \xrightarrow{\text{mono } g} A$, $E \xrightarrow{\text{mono } f} ES$ there exists

$ES \xrightarrow{\text{mono } \omega} A$ such that the diagram



commutes. That is $f = \omega g$.

Now $f = \omega g \Rightarrow f = \omega\varphi f$

So $I = \omega\varphi$

Again $g = \varphi f \Rightarrow g = \varphi\omega g$

So $I = \varphi\omega$

Thus ω and φ both are invertible which implies both ω and φ are isomorphic.

Hence $ES \cong A$ □

Definition 4.2.5: A near-ring N is said to be a regular near-ring if for every element $x \in N$, there exists an element $y \in N$ such that $xyx = x$.

Theorem 4.2.6: Let E be quasi-injective N-group let $\Lambda = \text{Hom}(E, E)$ and let $J = J(\Lambda)$ denote the Jacobson radical of Λ and is strictly small in Λ. Then

$J = \{ \lambda \in \Lambda / E \supseteq_e \text{Ker}\, \lambda \}$. If for $\gamma \in J$, $\lambda \in \Lambda$, $\gamma\lambda \in J$ then Λ / J is a regular near ring.

Where addition of two N-subgroups is again N-subgroup of E and N need not be dgnr.($\supseteq_e$ denotes essential extension)

Proof: Let $I = \{\lambda \in \Lambda / E \supseteq_e \text{Ker}\, \lambda\}$

If $\lambda \in \Lambda$, $\mu, \gamma \in I$, then $\text{Ker}(\mu + \gamma) \supseteq \text{Ker}\, \mu \cap \text{Ker}\, \gamma$

Since $x \in \text{Ker}\, \mu \cap \text{Ker}\, \gamma \Rightarrow x \in \text{Ker}\, \mu$ & $x \in \text{Ker}\, \gamma \Rightarrow \mu(x) = 0$ & $\gamma(x) = 0$

$\Rightarrow (\mu + \gamma)(x) = 0 \Rightarrow x \in Ker(\mu + \gamma)$

Since Ker$\mu \cap$ Ker γ is an essential N-subgroup.

Therefore Ker $(\mu + \gamma)$ is essential N-subgroup of E.

$x \in Ker\,\gamma \Rightarrow \gamma(x) = 0.$

Now for $\mu, \lambda \in \Lambda$, $\gamma \in I$, $(\mu(\lambda + \gamma) - \mu\lambda)(x) = (\mu(\lambda + \gamma))(x) - (\mu\lambda)(x)$

$= (\mu\lambda)(x) + 0 - (\mu\lambda)(x) = 0$ [since $\gamma(x) = 0$.] .

$\therefore x \in Ker(\mu(\lambda + \gamma) - \mu\lambda)$. And so Ker $\gamma \subseteq Ker(\mu(\lambda + \gamma) - \mu\lambda)$.

$\Rightarrow (\mu(\lambda + \gamma) - \mu\lambda) \in I.$

$\therefore$ I is left ideal of Λ.

However if $\lambda \in I$, $Ker(1 + \mu\lambda) = 0$ for $Ker\,\lambda \cap Ker(1 + \mu\lambda) = 0$.

For if, $\lambda \in I$ we have $E \supseteq_e Ker\,\lambda$. $x \in Ker\,\lambda \cap Ker(1 + \mu\lambda) \Rightarrow \lambda(x) = 0$ and $(1 + \mu\lambda)(x) = 0$

$\Rightarrow x + \mu(\lambda(x)) = 0 \Rightarrow x + \mu(0) = 0 \Rightarrow x = 0$. Again $\lambda \in I \Rightarrow E \supseteq_e Ker\,\lambda \Rightarrow Ker(1 + \mu\lambda) = 0$.

$1 + \mu\lambda : E \to (1 + \mu\lambda)E$ is an isomorphism $\Rightarrow \exists\, g \in \Lambda$ such that $g(1 + \mu\lambda) = I$, so $(1 + \mu\lambda)$ has a left inverse $\forall\, \lambda \in I$ & $\forall\, \mu \in \Lambda$.

So $\lambda \in J$ [by theorem 4.1.6].

This establishes that $I \subseteq J$.

Next let λ be arbitrary element of Λ, let L be a complement N-subgroup of E corresponding to $K = Ker(\lambda)$ and consider the correspondence $\lambda x \to x \quad \forall\, x \in L$.

If $\lambda x = \lambda y$ with $x, y \in L$, then $\lambda(x - y) = 0$ and then $x - y \in K \cap L = 0$

Since E is quasi-injective, the map $\lambda x \to x$ of λL in L is induced by some $\theta \in \Lambda$.

If $u = x + y \in L + K$, $x \in L$, $y \in K$, then

$(\lambda - \lambda\theta\lambda)(u) = \lambda(x) - \lambda\theta\lambda(x) = \lambda(x) - \lambda(x) = 0$

$\Rightarrow \lambda - \lambda\theta\lambda = 0$ .....*** $[\lambda\theta\lambda(x) = \lambda\theta(x) = \lambda(x) = x$, as for $x \in L$ $\theta(x) = \lambda(x) = x]$

Since $E \supseteq_e L + K$ {as $K \subseteq L + K$} and since $Ker(\lambda - \lambda\theta\lambda) \supseteq L + K$, we conclude that $\lambda - \lambda\theta\lambda \in I$.

Also I is an ideal. Thus Λ is a regular modulo I.

Now to show J = I. If $\lambda \in J$ and $\theta \in \Lambda$ is chosen so that $u = (\lambda - \lambda\theta\lambda) \in I$, $(1 - \theta\lambda)^{-1}$ exists. (Since J is Jacobson redical)

Therefore $(1 - \theta\lambda)^{-1}u = (1 - \theta\lambda)^{-1}(\lambda - \lambda\theta\lambda) = (1 - \theta\lambda)^{-1}(1 - \theta\lambda)\lambda = \lambda$ and $\lambda \in I$ [$\because$ I is a left ideal]. Thus J = I is as asserted.

From *** $\overline{\lambda\theta\lambda} = \bar{\lambda}$ in Λ/I.

$\therefore$ Λ/I is regular ring. $\therefore$ Λ/J is regular ring.

4.3 SOME PROPERTIES OF QUASI-INJECTIVE N-GROUPS:

This section contains some properties of quasi-injective N-groups related to essentially closed N-subgroups and complement N-subgroups. In this section we attempt to study various characteristics of quasi-injective N-groups satisfying chain conditions. In the third chapter we have investigated various characteristics of N-groups satisfying ascending chain condition on essential ideals and also investigated almost weakly Noetherian N-groups. Using the results proved in chapter 3, we try to establish new relations in quasi-injective N-groups satisfying chain conditions.

Let M be an N-subgroup of E.

We consider $F = \{ P/ P$ N-subgroup of E, $P \cap M = 0\}$

$F \neq \phi$, $(0) \in F$

$C = \{P_i / P_i \in F\}$ is a chain in F.

Let $K = \cup P_i$ [$x, y \in \cup P_i \Rightarrow x \in P_i, y \in P_j$

If $i > j$, $x, y \in P_j$ $\therefore x-y \in P_j \Rightarrow x-y \in \cup P_i$.

Again $n \in N$, $x \in \cup P_i \Rightarrow x \in P_j$ for some j, then $nx \in P_j \Rightarrow nx \in \cup P_i$]

$\because Pi \cap M = 0 \quad \forall i$

$(\cup_i Pi) \cap M = \cup_i (Pi \cap M) = 0$ & $\cup_i Pi \leq_S E$

$\therefore \cup_i Pi \in C$

So by Zorn's lemma the N-subgroup K is maximal in the set of those N-subgroups P satisfying

$P \cap N = 0$. Then K is said to be complement of M in E.

Definition 4.3.1: The N-subgroup K is maximal in the set of those N-subgroups P satisfying

$P \cap N = 0$ is said to be complement of M in E.

A complement N-subgroup (ideal) of E is a N-subgroup A which is a complement in E of some N-subgroup (ideal) B.

The following is an example of an N-group where sum of two N-subgroups is again an N-subgroup.

Example 4.3.2: N = {0, a,b,c} is the Klein's four group with multiplication

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	c	b	c

is a near-ring . Here A = {0, a}, B = {0, b}, C = {0, c} are the non-trivial N-subgroups of $_NN$ and sum of two N-subgroups is also an N-subgroup.

If sum of two N-subgroups is again an N-subgroup of an N-group we get following three lemmas and the corollary.

Lemma 4.3.3: If M is an N-subgroup of E and if K is any complement of M in E, then there exists a complement Q of K in E such that Q⊇M. Furthermore any such Q is a maximal essential extension of M in E.

Proof: Let F = { I/ I∩K = {0}, M ⊆ I } . Since M ∈ F. F ≠ φ

Let C = {C_i/i ∈ λ, λ index, C_i ∈ F} be a chain.

Q = ∪ C_i

Now $(\cup_{i\in\lambda} C_i) \cap K = \bigcup_{i\in\lambda}(C_i \cap k) = 0 \quad \forall i \quad [\because C_i \cap K = 0 \ \forall i]$

$\& M \subset \bigcup_{i\in\lambda} C_i \ \forall i, M \subseteq C_i$

So by Zorn's lemma Q∈F, maximal element exists. Thus Q in the first sentence exists.

Now to prove second assertion.

Let T be any non-zero N-subgroup of Q and assume that $T \cap M = 0$

Since $T \cap K = 0$ $[Q \leq_C K, T \leq_S Q]$

$\therefore$ the sum $K_1 = T + K$ is direct and K_1 properly contains K.

Since $K_1 \cap M = 0$ [If possible let $K_1 \cap M \neq 0$. $K_1 \cap M = (T + K) \cap M$

Let $t + k = n \in (T + K) \cap M \Rightarrow k \in K \cap (M + T) \subseteq K \cap Q \Rightarrow k = 0 \Rightarrow n = t \in M \cap T$

contradiction to $M \cap T = 0$. Therefore $K_1 \cap M = 0$]

This contradicts the definition of K.

This proves that Q is an essential extension of M.

If P is an N-subgroup of E properly containing Q, then $P \cap K \neq 0$ and

$(P \cap K) \cap M = P \cap (K \cap M) = P \cap 0 = 0.$

Thus P is not an essential extension of M, completing the proof.

Lemma 4.3.4: The essentially closed N-subgroups of an N-group E coincide with the complement N-subgroups of E. If M and K are complement N-subgroups and if K is a complement of M in E then M is a complement of K in E.

Proof: Let M be a essentially closed N-subgroup and let K is any complement of M. Then by lemma 4.3.3 there exists a complement Q of K such that $M \subseteq Q$. This Q is maximal essential extension of M in E. But M is essentially closed, so it has no proper essential extension.

$\therefore$ M = Q is a complement N-subgroup.

Next let M be complement of an N-subgroup P. Then $\exists$ a complement K of M which contains P.

i.e. $\overset{\max}{M} \cap P = (0)$ ....(1)

$\overset{\max}{K} \cap M = (0)$ such that $P \subset K$.

If possible let $M' \leq_S E$ such that $M \subseteq M'$ & $K \cap M' = (0)$

Then $P \cap M' = (0)$ $\because P \subset K$, which contradicts (1).

$\therefore$ M is also maximal such that. $K \cap M = (0)$. $\therefore$ M is complement of K. Then M is essentially closed by lemma 4.3.3.

This also proves the last statement.

Theorem 4.3.5: Let E be quasi-injective and let M be a essentially closed N-subgroup, then for each N-subgroup K of E, N-homomorphism $w : K \to M$ can be extended to N-homomorphism $u : E \to M$.

Proof: Let F = {L / w is extended to a map of T into M for $T \leq E$ containing L}

By Zorn's lemma we can assume that K is such that w cannot be extended to a map of T into M for any N-subgroup T of E which properly contains K.

Since E is quasi-injective, w is induced by a map $u : E \to E$ & let L complement of M in E.

Suppose $u(E) \not\subseteq M$.

Since M is essentially closed. M is a complement of L.

Therefore, since $u(E) + M \supset M$, we see that

$(u(E) + M) \cap L \neq 0$.

Let $0 \neq x = a + b \in u(E) + M) \cap L$

$a \in u(E), b \in M$

If $a \in M$ then $x \in M \cap L = 0$, a contradiction.

Therefore $a \notin M$ and $a = x - b \in L + M$

Now $T = \{y \in E / u(y) \in L + M\}$ is an N-subgroup of E containing K.

$\because x \in K \Rightarrow w(k) \in M \Rightarrow u(k) \in M \;\; \forall k \in K$.

$\therefore$ T contains K.

If $y \in E$ is such that $u(y) = a$ then $y \in T$, but $y \notin K$ since $a \notin M$.

[$\because y \in T \Rightarrow u(y) = a \in L$ & $y \in K \Rightarrow w(y) \in M \Rightarrow u(y) \in M \;\; \forall y \in K$, contradiction to $a \notin M$]

Let π denote the projection of L + M on M. Then πu is a map of T in M and

$\pi u(y) = u(y) = w(y) \;\; \forall y \in K$ [$\because y \in K \Rightarrow w(y) \in M \;\; \Rightarrow u(y) \in M \;\; \forall y \in K$]

Thus πu is a proper extension of w, a contradiction.

$\therefore u(E) \subseteq M$, so u is the desired extension.

Cor 4.3.6: Let E be quasi-injective N-group then

(1) If M is any essentially closed N-subgroup of E, then M is direct summand of E and M is quasi-injective. Also M has a complement in E.

(2) If P is any N-subgroup of E, then there exists a quasi-injective essential extension of P contained in E.

(3) Each minimal quasi-injective extension of an N-group K is an essential extension of K.

Proof: (1) If $e : E \rightarrow M$ is the extension given by the theroem4.3.5 of the injection map $M \rightarrow M$ then $E = M \oplus Ker(e)$ where $e(m) = \begin{cases} m & m \in M \\ 0 & m \notin M \end{cases}$

So that M is direct summand of E. ∴ M is quasi-injective by theorem 4.1.11.

Moreover Ker(e) is complement of M. Since $M \cap Ker(e) = (0)$ … … … (1)

M essentially closed ⇒ M complement of some N-subgroup K.

⇒ K is complement of M.

i.e. $\overset{max}{M} \cap \overset{max}{K} = (0)$………..(2)

(1)&(2) ⇒ $Ker(e) \subset K$

Let $(0 \neq) x \in K \Rightarrow x \notin M$

⇒ $e(x) = 0$ [by definition of e]

⇒ $x \in Ker(e)$.

∴ $K \subset Ker(e)$

∴ $Ker(e) = K \Rightarrow Ker(e)$ complement of M.

(2) Let $F = \{I / P \subseteq I, P \leq_e I\}$

$P \in F$. ∴ $F \neq \phi$

Let $\{ C_i / C_i \in F\}$ be a chain in F.

$M = \cup_i C_i$, $P \subseteq C_i \quad \forall i$

$\Rightarrow P \subseteq \cup_i C_i \quad P \leq_e C_i \quad \forall i$

$\Rightarrow P \leq_e \cup_i C_i$ [$P \cap A_i \neq 0 \ \forall i, A_i \leq C_i$. Since $P \cap (\cup_i A_i) = \cup_i (P \cap A_i) \neq 0$.

$\cup A_i \leq \cup C_i$]

If possible $M = \cup_i C_i \leq_e K$

$\therefore P \leq_e M \leq_e K \Rightarrow P \leq_e K$, contradicts maximality of M.

So by Zorn's lemma P is contained in essentially closed N-subgroup M which is essential extension of P and M is quasi-injective by (1).

(3) Let A be any minimal quasi-injective extension of an N-group K. Let K is contained in quasi-injective essential extension B by (2)

i.e. $K \leq_e B$, B essentially closed.

So $A \subseteq B$

As B is essential extension of K, A is also essential extension of K.

Thus every minimal quasi-injective extension of N-group K is an essential extension of K.

Definitions 4.3.7: An N-group E is said to have finite Goldie dimension if it does not contain an infinite direct sum of non-zero ideals of E.

For an N-group E if there exists an integer n such that E has an independent family of n non-zero ideals, but no independent families of more than n non-zero ideals, then integer n is called the Goldie dimension of E.

The proof of the following proposition follows the same line of proof as N. V. Dung [22].

Proposition 4.3.8: Let E be a finitely generated quasi-injective left N-group. Suppose E contains an infinite direct sum of non zero independent family of ideals $H = \oplus_{\lambda}H_{\lambda}$. Then the factor N- group E/H has infinite Goldie dimension.

Proof: Assume E/H has finite Goldie dimension k.

Since the index set Λ is infinite, we find infinite subsets $\Lambda_1, \Lambda_2, \ldots\ldots\ldots \Lambda_{k+1}$ such that

$\Lambda_i \cap \Lambda_j = 0$ for $i \neq j$ and $\Lambda = \Lambda_1 U \ldots\ldots\ldots U\, \Lambda_{k+1}$

For $S_j = \oplus_{\Lambda_j} H_{\lambda}$ $(J = 1, 2, \ldots\ldots\ldots, k+1)$ we get $H = S_1 \oplus S_2 \oplus \ldots\ldots\ldots \oplus S_{k+1}$

[Since $H_{\lambda} \leq_e S_j = \oplus_{\Lambda_j} H_{\lambda}$, for $\lambda \in \Lambda_j$, as $x_i \in H_{\lambda_i}$.

$x_i = 0 + 0 + \ldots\ldots\ldots + 0 + x_i + 0 + \ldots\ldots\ldots + 0 + 0 \in \oplus_{\Lambda_j} H_{\lambda}$.]

$\Rightarrow$ S_i's are independent, as H_i's are independent by proposition1.3.6.

$x \in H = \oplus_{\Lambda}H_{\lambda} \Rightarrow x = \sum_{i\in\lambda} x_i$ unique.

$= \sum_{i_1\epsilon\lambda_1} x_{i_1} + \sum_{i_2\epsilon\lambda_2} x_{i_2} + \ldots\ldots\ldots + \sum_{i_{k+1}\epsilon\lambda_{k+1}} x_{i_{k+1}}$

[$\because \Lambda = \Lambda_1 U \ldots\ldots\ldots U\, \Lambda_{k+1}, \Lambda_I \cap \Lambda_J = 0$ where $\sum_{i_n\epsilon\lambda_n} x_{i_n}$ unique]

$\in S_1 \oplus S_2 \oplus \ldots\ldots\ldots \oplus S_{k+1}$.

Then we have $E(H) = E(S_1) \oplus E(S_2) \oplus \ldots \oplus E(S_{k+1})$ by proposition1.4.13.

[$\because S_j \leq_e E(S_j)$ and $\{S_j\}$ independent $\Rightarrow E(S_j)$ independent by proposition1.3.6.]

$E(S_j) \neq S_j$ for each j.

[Note: Now for $H = S_1 \oplus S_2$,$E(H) = E(S_1) \oplus E(S_2)$ we show $\frac{E(H)}{H} \cong \frac{E(S_1)}{S_1} \oplus \frac{E(S_2)}{S_2}$,

where H, S_1, S_2 ideal of E.

Proof: Define a map $\phi : \frac{E(H)}{H} \longrightarrow \frac{E(S_1)}{S_1} \oplus \frac{E(S_2)}{S_2}$ by $\phi(x + H) = (s_1 + S_1, s_2 + S_2)$ where $x = (s_1, s_2) \in E(H)$ with $s_1 \in E(S_1)$, $s_2 \in E(S_2)$.

ϕ is well-defined:

Let $(x_1 + S_1, y_1 + S_2) \neq (x_2 + S_1, y_2 + S_2) \in \frac{E(S_1)}{S_1} \oplus \frac{E(S_2)}{S_2}$ where $(x_1, y_1) = h_1 \in E(H)$, $(x_2, y_2) = h_2 \in E(H)$.

To show $h_1 + H \neq h_2 + H$.

Now $(x_1 + S_1, y_1 + S_2) \neq (x_2 + S_1, y_2 + S_2)$

$\Rightarrow$ either $x_1 + S_1 \neq x_2 + S_1$ or $y_1 + S_2 \neq y_2 + S_2$ or both.

$\Rightarrow$ either $x_1 - x_2 \notin S_1$ or $y_1 - y_2 \notin S_2$ or both.

If possible let $h_1 + H = h_2 + H$

$\Rightarrow h_1 - h_2 \in H$

$\Rightarrow (x_1, y_1) - (x_2, y_2) \in H$

$\Rightarrow (x_1 - x_2, y_1 - y_2) \in H = S_1 \oplus S_2$

$\Rightarrow x_1 - x_2 \in S_1$ or $y_1 - y_2 \in S_2$, a contradiction, so $h_1 + H \neq h_2 + H$.

So ϕ is well-defined.

ϕ is 1-1:

Let $(\neq \bar{0})\ x_1 + H \neq x_2 + H \in \frac{E(H)}{H}$

i.e. $x_1, x_2 \in E(H)$ but $\notin H$ and $x_1 - x_2 \notin H$.

Let $x_1 = (s_{11}, s_{21})$, $x_2 = (s_{12}, s_{22})$ with $s_{11}, s_{12} \in E(S_1)$ and $s_{21}, s_{22} \in E(S_2)$.

$\phi(x_1 + H) = (s_{11} + S_1, s_{21} + S_2)$, $\phi(x_2 + H) = (s_{12} + S_1, s_{22} + S_2)$

If possible let $\phi(x_1 + H) = \phi(x_2 + H)$

$\Rightarrow (s_{11} + S_1, s_{21} + S_2) = (s_{12} + S_1, s_{22} + S_2)$

i.e. $s_{11} + S_1 = s_{12} + S_1$ and $s_{21} + S_2 = s_{22} + S_2$

$\Rightarrow s_{11} - s_{12} \in S_1$ and $s_{21} - s_{22} \in S_2$

But since $x_1 - x_2 \notin H \Rightarrow (s_{11}, s_{21}) - (s_{12}, s_{22}) \notin H$

$\Rightarrow (s_{11} - s_{12}, s_{21} - s_{22}) \notin H = S_1 \oplus S_2$

$\Rightarrow$ either $s_{11} - s_{12} \notin S_1$ or $s_{21} - s_{22} \notin S_2$ or both, which is a contradiction.

So $\phi(x_1 + H) \neq \phi(x_2 + H)$ i.e. ϕ is 1-1.

ϕ is onto:

Let $(x', y') \in \frac{E(S_1)}{S_1} \oplus \frac{E(S_2)}{S_2}$

$\Rightarrow x' \in \frac{E(S_1)}{S_1}, y' \in \frac{E(S_2)}{S_2}, x' \neq y'$

$\Rightarrow \exists\ s_1 \in E(S_1), s_2 \in E(S_2)$ such that $s = (s_1, s_2) \in E(S_1) \oplus E(S_2) = E(H)$

$x' = (s_1 + S_1), y' = (s_2 + S_2)$ and $\phi(s + H) = (s_1 + S_1, s_2 + S_2) = (x', y')$.

$\therefore$ ϕ is onto. So ϕ is isomorphism.]

Similarly as above note we get the isomorphism

$\frac{E(H)}{H} \cong \frac{E(S_1)}{S_1} \oplus \frac{E(S_2)}{S_2} \oplus \ldots\ldots\ldots \oplus \frac{E(S_{k+1})}{S_{k+1}}$ which is a ideal of $\frac{E}{H}$.

The Goldie dimension of $\frac{E}{H}$ is at least k+1, a contradiction.

Thus $\frac{E}{H}$ has infinite Goldie dimension.

Theorem 4.3.9: Let N be dgnr and E be a quasi- injective, finitely generated N-group. If E has A.C.C. on essential ideals, then E is weakly Noetherian.

Proof: Assume E has A.C.C. on essential ideals. Then by proposition 3.4.10 $\frac{E}{Soc(E)}$ is weakly Noetherian. So $\frac{E}{Soc(E)}$ cannot contain an infinite direct sum of ideals. i.e. $\frac{E}{Soc(E)}$ has finite Goldie dimension. So by proposition 4.3.8, E cannot contain an infinite direct sum SocE = $\oplus_\lambda M_\lambda$. i.e. SocE is finite direct sum of simple ideals. Since every simple ideal is weakly Noetherian, by corollary 4.1.8 SocE is weakly Noetherian. Now if we consider the exact sequence

$0 \longrightarrow SocE \longrightarrow E \longrightarrow \frac{E}{Soc(E)} \longrightarrow 0$, SocE and $\frac{E}{Soc(E)}$ are weakly Noetherian, so by proposition 4.1.7 E is also weakly Noetherian.

For near-ring N with identity and M unital N-group if for every right ideal U of N and every N-homomorphism f : U ⟶ M, there exists an element m in M such that f(a) = ma for all a in U implies M is injective then we get the following two results.

Proposition 4.3.10: Let N be a near ring with unity and let E be a unital N-group.Then the direct sum $Q = N \oplus E$ is a quasi-injective N-group if and only if both N and E are injective.

Proof: The sufficiency is trivial. [N, E injective $\Rightarrow N \oplus E$ injective $\Rightarrow N \oplus E = Q$ quasi-injective(as injective $\Rightarrow$ quasi-injective)]

Conversely if Q is quasi-injective, so are N and E since direct summands of quasi-injective N-group are quasi-injective. As N is quasi-injective, for every right ideal U of N and every N-homomorphism $f : U \longrightarrow N$, there exists an element m in N such that f(a) = ma for all a in U. So N is injective.

Let f be any map of an ideal I of N into E. Writing the elements of Q as ordered pairs (n, e), $n \in N$, $e \in E$ the correspondence $(x,0) \rightarrow (0, f(x))$ defined for all $x \in I$ is a map of an N-subgroup of Q into Q and therefore has an extension $f' : Q \rightarrow Q$.

Set $f'(1,0) = (s,n)$

If $x \in I$ then $(0, f(x)) = f'(x,0) = f'(1,0)x = (sx, nx)$ i.e. $f(x) = nx \;\; \forall x \in I$.

Thus E is injective.

Following is the corollary of theorem 4.2.6 with those same conditions.

Corollary 4.3.11: If J = 0 , Λ is a regular near-ring. Moreover if regular near-ring Λ is such that each finitely generated ideal of Λ is generated by idempotent, then Λ_Λ is injective.

Proof: Λ is a regular Near-ring .

Let $f : I \rightarrow \Lambda$ be any map of a ideal I into Λ. By IE we mean that N-subgroup of E generated by $\{ \lambda m / \lambda \in I, m \in E\}$ and it follows that if $x \in IE$ then there exists m_1, m_2, m_3, ... , $m_n \in E$, $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots \ldots, \lambda_n \in I$ such that such that $x = \sum_{i=1}^{n} \lambda_i m_i$. We consider a correspondence

$$x = \sum_{i=1}^{n} \lambda_i m_i \rightarrow \sum_{i=1}^{n} f(\lambda_i) m_i$$

If also $y = \sum_{j=1}^{t} \mu_j m_j' \in IE$, $\mu_j \in I$, $m_j' \in E$, $j = 1, 2, 3, \ldots \ldots \ldots, t$ then the ideal generated by $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots \ldots, \lambda_n, \mu_1, \mu_2, \mu_3, \ldots \ldots \ldots, \mu_n$ has the form $e\Lambda$, where $e = e^2 \in \Lambda$ and then $e\lambda_i = \lambda_i$, $e\mu_j = \mu_j$.

$$f(\lambda_i) = f(e)\,\lambda_i\,,\ f(\mu_j) = f(e)\,\mu_j\,,\ i = 1,2,3, \ldots \ldots \ldots n,\ j = 1,2,3, \ldots \ldots \ldots, t$$

Consequently $\sum_{i=1}^{n} f(\lambda_i) m_i = \sum_{i=1}^{n} f(e)\lambda_i m_i = f(e) \sum_{i=1}^{n} \lambda_i m_i = f(e)x$,

Similarly, $\sum_{j=1}^{t} f(\mu_j) m_j' = f(e)y$, so that $x \rightarrow \sum_{i=1}^{n} f(\lambda_i) m_i$ is a single valued correspondence, θ is a map of IE in E. By quasi-injectivity θ is induced by an element of Λ, which is also denoted by θ. Then $(\theta\lambda)(m) = \theta(\lambda m) = f(\lambda)(m)$ $\forall \lambda \in I, m \in E$ so that $f(\lambda) = \theta \lambda$) $\forall \lambda \in I$.

So we conclude that Λ_λ is injective.

4.4. SUPERHONESTY IN QUASI-INJECTIVE N-GROUPS:

In chapter 2 we have studied many properties and characteristics of superhonest near-ring groups. In this section we try to establish some characteristics of superhonest quasi-injective near-ring groups.

Theorem 4.4.1 : : If E is a quasi-injective N-group, $Cl_\chi\, T_\chi(E)$ is super-honest in E implies $Cl_\chi\, T_\chi(E) = P$, the smallest super-honest N-subgroup of E.

Proof: Since P contains $T_N(E)$, so it contains $T_\chi(E)$. [Since every superhonest N-subgroup contains $T_N(E)$ by note 2.3.18(2) and $T_N(E)$ contains $T_\chi(E)$]

Then P is χ-closed N-subgroup of E, therefore $P = Cl_\chi\,(P) \supset Cl_\chi\, T_\chi(E)) = T_\chi\, T_\chi(E)$. [Since $Cl_\chi\, T_\chi(E) = T_\chi\, T_\chi(E)$ by proposition 2.3.23]

Also $T_\chi\, T_\chi(E) \supset P$, since P is the smallest superhonest N-subgroup of E.

Hence $Cl_\chi\, T_\chi(E) = P$.

If sum of two N-subgroups is again an N-subgroup of an N-group then we get the following four results:

Theorem 4.4.2 If M is a χ-closed ideal of an quasi-injective N-group E and $T_N(E) \subseteq M$ then M is super-honest ideal of E .

Proof: Let $a \in E \setminus M$ with $na \in M$ for some $n \in N$.

Since M is a χ-closed ideal of E, by corollary 4.3.6(i), $E = M \oplus M^c$, where M^c is a complement N-subgroup of M in E, since M is χ-closed implies essentially closed.

Then $a = m + m'$ for some $m \in M$ and $m' \in M^c$.

Now $m' = -m + a$ implies $n(-m + a) = n(-m + a) - na + na = nm' \in M \cap M^c = 0$.

But $0 \neq m'$ (for otherwise $a = m \in M$) and $m' \notin T_N(E)$ (Since $M \supset T_N(E)$ and $a \notin M$), so n = 0. Hence M is super-honest in E.

Theorem 4.4.3: If E is a quasi-injective N-group and $T_\chi\, T_\chi(E)$ is ideal of E containing $T_N(E)$ then $T_\chi\, T_\chi(E)$ is super-honest in E.

Proof Since $T_\chi\, T_\chi(E)$ is a χ-closed ideal of E and $T_\chi\, T_\chi(E) \supset T_N(E)$, $T_\chi\, T_\chi(E)$ is super-honest in E by theorem 4.4.2.

Note 4.4.4: From corollary 2.3.21 we know that if P is the smallest super-honest N-subgroup of an N-group E then $f(P) \subseteq P$ for each N-endomorphism f of E.

Theorem 4.4.5: If E is a quasi-injective N-group with non trivial super-honest N-subgroups P′ is a proper N-subgroup of E such that $P' \supset P$ then there exists an N-endomorphism f of E such that $f(P') \not\subset P'$.

Proof: Let $a \in E - P'$, $b \in P' - P$. Since $P' \supset P \supset T(E)$, a and b are both not in $T_N(E)$.

Then we have an N-homomorphism ϕ from Nb to Na which is defined by $\phi(nb) = na$ for each $n \in N$.

Since, E is quasi-injective ϕ can be extended to an N-endomorphism f on E.

But then $a \in Na = f(Nb) \subseteq f(P')$. Since $a \notin P'$, we have $f(P') \subseteq P$.

For the following results we assume the ideal character of χ-closure of the N-subgroup generated by $T_N(E)$.

Theorem 4.4.6: If E is an quasi-injective N-group, then the smallest super-honest N-subgroup $P = Cl_\chi(D)$, where $Cl_\chi(D)$ is the χ-closure of the N-subgroup D generated by $T_N(E)$.

Proof: Since every super-honest N-subgroup of E contains $T_N(E)$(hence contains D) and is a χ-closed N-subgroup of E. [by 2.3.13 and 2.3.11]

We have $P = Cl_\chi(P) \supseteq Cl_\chi(D)$.

On the otherhand since $T_\chi(E) \subseteq T_N(E) \subseteq D \subseteq Cl_\chi(D)$, by lemma 2.3.10, $Cl_\chi(D)$ is an essential N-subgroup of $Cl_\chi\, Cl_\chi(D)$ and D is an essential N-subgroup of $Cl_\chi(D)$.

Therefore D is an essential N-subgroup of $Cl_\chi\, Cl_\chi(D)$.

Then $Cl_\chi\, Cl_\chi(D) \subseteq Cl_\chi(D)$ [by proposition 2.3.23].

So $Cl_\chi\, Cl_\chi(D) = Cl_\chi(D)$ is χ-closed in E.

Since $Cl_\chi(D) \supseteq T_N(E)$ by proposition 4.4.2, $Cl_\chi(D)$ is super-honest ideal in E.

Hence $Cl_\chi(D) \supseteq P$.

$\therefore\ Cl_\chi(D) = P$.

Theorem 4.4.7 : If E is a semi-simple quasi-injective N-group then the smallest super-honest N-subgroup P = D where D is the N-subgroup of E generated by $T_N(E)$ and $Cl_\chi(D)$ is an ideal of E.

Proof: Since E is a semi-simple, every ideal of E is a direct summand of E.

Again, E is quasi-injective, so $P = Cl_\chi(D)$ by theorem 4.4.6.

But every N-subgroup of an semi-simple N-group is semi-simple. Thus D is direct summand of $Cl_\chi(D)$ and is an essential N-subgroup of $Cl_\chi(D)$. Hence $D = Cl_\chi(D) = P$.

4.5 SOME RELATIONS OF QUASI-INJECTIVITY WITH RELATIVE INJECTIVITY:

In this section we attempt to find some relations between weak singular quasi-injective N-groups and relative injective N-groups.

Definition4.5.1: A near-ring N is called QI-near-ring if every quasi-injective N-group is injective.

Theorem 4.5.2: For a dgnr S^3I-near-ring N, if every injective right N/K-group is injective as an N-group for ideal K of N we get the following conditions equivalent:

i. Every weak singular quasi- injective N-group is injective.

ii. Z(N) = 0 and direct sum of weak singular quasi-injective N-groups is injective.

Proof: i.⇒ii. Since N is S^3I-near-ring, Z(N) = 0 by lemma 3.4.21.

From proposition 3.4.25, direct sum of weak singular injective N-groups is injective. From given condition we get direct sum of weak singular quasi-injective N-groups is injective.

ii.⇒i. Let A be a weak singular quasi-injective N-group. E(A) is weak singular as proposition 3.4.17. By hypothesis A ⊕ E(A) is injective. So A is injective.

Theorem 4.5.3. For a near-ring N, $^{N}/_{Soc(N)}$ is a QI-near-ring implies every singular quasi-injective N-group is injective.

Proof: If E is singular quasi-injective N –group then Soc(N). E = 0 and so E is quasi-injective $^{N}/_{Soc(N)}$-group. Whence E is injective as an $^{N}/_{Soc(N)}$-group. So E is injective as an N-group.

Theorem 4.5.4: If every singular quasi-injective N-group is injective if and only if N is QI-ring then $N/_{Soc(N)}$ is a QI-near-ring.

Proof: If E is singular quasi-injective $N/_{Soc(N)}$-group, then E is singular quasi-injective N –group. Thus E is injective as an N-group hence injective as an $N/_{Soc(N)}$-group. Whence $N/_{Soc(N)}$ is a near-ring all of whose singular quasi-injective $N/_{Soc(N)}$-groups are injective. So, $N/_{Soc(N)}$ is a QI-near-ring.